

Quantum Cloning and Deletion in Quantum Information Theory

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DECLARATION

I declare that the thesis entitled “*Quantum Cloning and Deletion in Quantum Information Theory*” is composed by me and that no part of this thesis has formed the basis for the award of any Degree, Diploma, Associateship, Fellowship or any other similar title to me.

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This is to certify that the thesis entitled “*Quantum Cloning and Deletion in Quantum Information Theory*” submitted by Satyabrata Adhikari who got his name registered on 21.12.2002 for the award of Ph.D.(Science) degree of Bengal Engineering and Science University, is absolutely based upon his own work under my supervision in the Department of Mathematics, Bengal Engineering and Science University, Shibpur, Howrah-711103, and that neither this thesis nor any part of it has been submitted for any degree / diploma or any other academic award anywhere before.

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Preface

Quantum ideas when incorporated into the domains of computation and information science have revolutionized the subject. In terms of potential performance it is recognized that quantum computation and information theories are far richer than their classical counterparts. There are two important results of quantum mechanics, namely impossibility of creating exact copies of quantum states, commonly known as 'no-cloning theorem' and the impossibility of deleting one of the two given identical quantum states commonly known as 'no-deletion principle' have simultaneously enhanced the scope of such theories in certain respects and restricted the theories in other directions. This thesis is concerned with some aspects of quantum cloning and deletion and their implications in quantum computation and information problems. The thesis is organized into five chapters. In the following we state the chapter wise summaries of the thesis:

In chapter-1, general introduction is given. In this introduction we discuss about various existing quantum cloning machines and quantum deletion machines.

In chapter-2, we discuss the combination of independently existing quantum cloning machines known as hybrid quantum cloning machine. In this chapter we study the state dependent and state independent hybrid quantum cloning machines.

Chapter-3 deals with the cloning of entanglement using local quantum cloners. This concept is popularly known as broadcasting of entanglement. We discuss the broadcasting of two-qubit entanglement using state dependent quantum cloners. Further we find that here that the broadcasting of three-qubit entanglement is also possible.

In chapter-4, we investigate about the universal quantum deletion machine. We show that universal quantum deletion machine exists with better fidelity of deletion only if an additional unitary operator called transformer is added with the unitary operator called

deleter.

In chapter-5, we study the effect on an arbitrary qubit as a result of concatenation of two quantum operations viz. unitary quantum cloning and deleting transformations.

Lastly, the list of publications are given and the list of references are given under the broad heading 'Bibliography'. We also have attached copies of our five published and one communicated works on which the thesis is based.

Contents

1	General Introduction	1
1.1	<i>Entanglement: A Non-Local Resource</i>	1
1.1.1	<i>What is entanglement?</i>	1
1.1.2	<i>Pure state entanglement</i>	2
1.1.3	<i>Measure of Pure state entanglement</i>	3
1.1.4	<i>Mixed state entanglement</i>	3
1.1.5	<i>Measure of Mixed state entanglement</i>	4
1.1.6	<i>Entanglement swapping</i>	7
1.2	<i>Distance measure between quantum states</i>	7
1.2.1	<i>Fidelity</i>	7
1.2.2	<i>Hilbert-Schmidt (H-S) distance</i>	8
1.3	<i>No-Cloning Theorem: A Brief History</i>	9
1.3.1	<i>Wigner's prediction</i>	9
1.3.2	<i>Herbert's Argument</i>	9
1.3.3	<i>Wootters and Zurek No-cloning Theorem</i>	11
1.4	<i>Description of quantum cloning machines</i>	14
1.4.1	<i>State dependent quantum cloning machines</i>	14
1.4.2	<i>State independent (Universal) quantum cloning machine</i>	20
1.4.3	<i>Probabilistic cloning</i>	30
1.4.4	<i>Phase covariant quantum cloning</i>	32
1.4.5	<i>Economical quantum cloning</i>	37
1.4.6	<i>Asymmetric quantum cloning</i>	39
1.5	<i>Quantum cloning of mixed state</i>	45

1.6	<i>Quantum cloning and no-signalling</i>	47
1.7	<i>Quantum Deletion machine</i>	49
2	Hybrid quantum cloning	55
2.1	<i>Prelude</i>	55
2.2	<i>State dependent hybrid cloning transformation</i>	59
2.2.1	<i>Hybridization of two B-H type cloning transformation:</i>	60
2.2.2	<i>Hybridization of B-H type cloning transformation and phase-covariant quantum cloning transformation</i>	64
2.3	<i>State independent hybrid cloning transformation</i>	66
2.3.1	<i>Hybridization of two BH type cloning transformations</i>	66
2.3.2	<i>Hybridization of optimal universal symmetric B-H cloning transformation and optimal universal asymmetric Pauli cloning transformation</i>	67
2.3.3	<i>Hybridization of universal B-H cloning transformation and universal anti-cloning transformation</i>	70
3	Broadcasting of entanglement	73
3.1	<i>Prelude</i>	73
3.2	<i>State dependent B-H quantum cloning machine</i>	77
3.3	<i>Broadcasting of entanglement using state dependent B-H quantum cloning machine</i>	81
3.4	<i>Secret broadcasting of 3-qubit entangled state between two distant partners</i>	87
3.5	<i>Discussion of quantum cryptographic scheme based on orthogonal states</i>	88
3.6	<i>Secret generation of two 3-qubit entangled state between three distant partners</i>	95
4	On universal quantum deletion transformation	102
4.1	<i>Prelude</i>	102
4.2	<i>Quantum deletion machines</i>	104
4.3	<i>Conventional deletion machine (deletion machine without transformer)</i>	106
4.4	<i>Modified deletion machine (deletion machine with single transformer)</i>	109

4.5	<i>Quantum deletion machine with two transformers</i>	113
4.6	<i>PB deleting machine with transformer</i>	115
5	Concatenation of quantum cloning and deletion machines	118
5.1	<i>Prelude</i>	118
5.2	<i>State dependent quantum deletion machine</i>	119
5.3	<i>Concatenation of cloning and deletion machines</i>	122
5.3.1	<i>Concatenation of WZ cloning machine and PB deleting machine</i>	122
5.3.2	<i>Concatenation of BH cloning machine and PB deleting machine</i>	123
5.3.3	<i>Concatenation of WZ cloning machine and deleting machine(5.1-5.4)</i>	124
5.3.4	<i>Concatenation of BH cloning machine and deleting machine(5.1-5.4)</i>	124

Chapter 1

General Introduction

All of modern physics is governed by that magnificent and thoroughly confusing discipline called quantum mechanics ... It has survived all tests and there is no reason to believe that there is any flaw in it.... We all know how to use it and how to apply it to problems; and so we have learned to live with the fact that nobody can understand it -
Murray Gell-Mann

1.1 *Entanglement: A Non-Local Resource*

1.1.1 *What is entanglement?*

In 1935, Einstein, Podolsky and Rosen (EPR) [56] presented a paradox that still surprises us today. Consider two physical systems that once interacted but are remote from each other now and do not interact. The two systems are still entangled if their quantum state does not factor into a product of states of each system. Entangled particles have correlated properties, and these correlations are at the heart of the paradox. Entanglement between quantum systems is a purely quantum mechanical phenomenon. It is closely related to the superposition principle and describes correlation between quantum systems that are much stronger and richer than any classical correlation could be. Mathematically entanglement can be defined in a following way:

Let us consider a system consisting of two subsystems where each subsystem is associated with a Hilbert space. Let H_A and H_B denote these two Hilbert spaces. Let $|i\rangle_A$ and $|j\rangle_B$ (where $i,j=1,2,3,\dots$) represent two complete orthonormal basis for H_A and H_B

respectively. The two subsystems taken together is associated with the Hilbert space $H_A \otimes H_B$, spanned by the states $|i\rangle_A \otimes |j\rangle_B$. Any linear combination of the basis states $|i\rangle_A \otimes |j\rangle_B$ is a state of the composite system AB and any pure state $|\psi\rangle_{AB}$ of the system can be written as

$$|\psi\rangle_{AB} = \sum_{i,j} c_{ij} |i\rangle_A \otimes |j\rangle_B \quad (1.1)$$

where the c_{ij} s are complex coefficients and $\sum_{i,j} |c_{ij}|^2 = 1$.

If $|\psi\rangle_{AB}$ factors into a normalized state $|\psi\rangle_A = \sum_i^{k=\dim(H_A)} c_i |i\rangle_A$ in H_A and a normalized state $|\psi\rangle_B = \sum_i^{k=\dim(H_B)} c_j |j\rangle_B$ in H_B , i.e. $|\psi\rangle_{AB} = |\psi\rangle_A \otimes |\psi\rangle_B$, then the state $|\psi\rangle_{AB}$ is called separable state or product state.

If a state belonging to the Hilbert space $H_A \otimes H_B$ is not a product state, then such a state is called entangled state.

1.1.2 Pure state entanglement

If $|\psi\rangle_{AB}$ represents a pure state of a composite system consisting of two Hilbert spaces H_A and H_B , then $|\psi\rangle_{AB}$ can always be written in Schmidt form (Schmidt decomposition) [111] as

$$|\psi\rangle_{AB} = \sum_i^{k \leq \min\{\dim H_A, \dim H_B\}} \sqrt{\lambda_i} |i\rangle_A \otimes |i\rangle_B \quad (1.2)$$

where $|i\rangle_A, |i\rangle_B$ are two orthonormal bases of systems A and B respectively and $\lambda_i \geq 0$, $\sum \lambda_i = 1$. The non-negative real numbers λ_i are known as Schmidt coefficients. If two or more Schmidt coefficients are non-zero, then the state $|\psi\rangle$ is referred to as Pure entangled state. If only one Schmidt coefficient is non-zero and all others are zero, then the state $|\psi\rangle$ is called product state. In particular, two qubit pure state can be written in the schmidt form as

$$|\psi\rangle_{AB} = \sum_{i=1}^2 \sqrt{\lambda_i} |i\rangle_A \otimes |i\rangle_B \quad (1.3)$$

where the Schmidt coefficients λ_1, λ_2 satisfy the normalization condition i.e. $\lambda_1 + \lambda_2 = 1$.

It has been shown that every pure entangled state violates some Bell-type inequality [78],

while no product state does. Entangled states cannot be prepared from unentangled states by any sequence of local actions of two distant partners, even with the help of classical communication. The most familiar example of pure entangled state is the singlet state of two spin- $\frac{1}{2}$ particles

$$|\psi^-\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \quad (1.4)$$

which cannot be reduced to direct product by any transformation of the bases pertaining to each one of the particles.

1.1.3 Measure of Pure state entanglement

1. Entropy of entanglement [16]: Let Alice(A) and Bob(B) share a pure entangled state $|\psi\rangle_{AB}$. Quantitatively, a pure state's entanglement is conveniently measured by its entropy of entanglement,

$$E|\psi_{AB}\rangle = S(\rho_A) = S(\rho_B) \quad (1.5)$$

Here $S(\rho) = -\text{Tr}(\rho \log_2 \rho)$ is the von-Neumann entropy and $\rho_A = \text{Tr}_B(|\psi\rangle_{AB}\langle\psi|)$, $\rho_B = \text{Tr}_A(|\psi\rangle_{AB}\langle\psi|)$ denote the reduced density matrices obtained by tracing the whole system's pure-state density matrix $|\psi\rangle_{AB}\langle\psi|$ over Bob's and Alice's degrees of freedom respectively.

1.1.4 Mixed state entanglement

Due to decoherence effect we usually deal with mixed states. A mixed state of quantum system consisting of two subsystems is supposed to represent entanglement if it is inseparable [16, 92, 96, 100, 114, 144] i.e. cannot be written in the form

$$\rho = \sum_i p_i (\rho_i^A \otimes \rho_i^B), \quad p_i \geq 0, \quad \sum_i p_i = 1 \quad (1.6)$$

where ρ_i^A and ρ_i^B are states for the two subsystems A and B respectively. A test for separability of (2×2) systems is the Peres-Horodecki criterion [91, 123], which states that a necessary and sufficient condition for the state $\hat{\rho}$ of two spins to be inseparable is that at least one of the eigen values of the partially transposed operator defined as

$\rho_{m\mu,n\nu}^{T_2} = \rho_{m\nu,n\mu}$ is negative. This is equivalent to the condition that at least one of the two determinants

$$W_3 = \begin{vmatrix} \rho_{00,00} & \rho_{01,00} & \rho_{00,10} \\ \rho_{00,01} & \rho_{01,01} & \rho_{00,11} \\ \rho_{10,00} & \rho_{11,00} & \rho_{10,10} \end{vmatrix} \quad \text{and} \quad W_4 = \begin{vmatrix} \rho_{00,00} & \rho_{01,00} & \rho_{00,10} & \rho_{01,10} \\ \rho_{00,01} & \rho_{01,01} & \rho_{00,11} & \rho_{01,11} \\ \rho_{10,00} & \rho_{11,00} & \rho_{10,10} & \rho_{11,10} \\ \rho_{10,01} & \rho_{11,01} & \rho_{10,11} & \rho_{11,11} \end{vmatrix} \quad \text{is negative}$$

$$\text{and } W_2 = \begin{vmatrix} \rho_{00,00} & \rho_{01,00} \\ \rho_{00,01} & \rho_{01,01} \end{vmatrix} \quad \text{is non-negative.}$$

We will use these conditions for inseparability in the subsequent chapter.

1.1.5 Measure of Mixed state entanglement

The fundamental law of quantum information processing says that the mean entanglement cannot be increased under local operation and classical communication (LOCC). The law actually says that there is some probability with which the two distant partners can obtain more entangled state. Then, however, with some other probability they will obtain less entangled states so that on average the mean entanglement will not increase. Under LOCC operations, one can only change the form of entanglement. It is known that concentration of entanglement is possible using local operation and classical communication. Therefore, to measure the efficiency with which one can perform this concentration, some measures of entanglement is introduced. Entanglement measures answer the following question 'how much entanglement is needed to create a given quantum state by local operation and classical communication alone?' or inversely 'how many singlets one can prepare from a supply of non-maximally entangled states?'. Now we have listed below the conditions which every 'decent' measure of entanglement should satisfy.

C1: The measure of entanglement for any separable state should be zero, i.e. $E(\rho) = 0$.

C2: The amount of entanglement in any state ρ should be unaffected for any local unitary transformation of the form $U_A \otimes U_B$, i.e. $E(\rho) = E(U_A \otimes U_B \rho U_A^\dagger \otimes U_B^\dagger)$.

C3: Local operations, classical communication and sub-selection cannot increase the expected entanglement i.e. $E(\rho) \geq \sum_i p_i E(\rho_i)$, where p_i denotes the probability with which the state ρ_i occurs.

C4: For any two given pairs of entangled particles in the total state $\rho = \rho_1 \otimes \rho_2$, we should have $E(\rho) = E(\rho_1) + E(\rho_2)$.

There are many measures of mixed state entanglement but in this thesis, we mention briefly just four of them.

1. Entanglement of formation [41, 64, 125, 148]: The entanglement of formation E_F of a bipartite mixed state ρ_{AB} is defined by

$$E_F(\rho_{AB}) = \min_{\rho_{AB} = \sum_i p_i |\psi_i\rangle_{AB} \langle \psi_i|} \sum_i p_i E(|\psi_i\rangle \langle \psi_i|) \quad (1.7)$$

The minimization in equation (1.7) is taken over all possible decompositions of the density operator ρ_{AB} into pure states $|\psi\rangle$. Entanglement of formation gives an upper bound on the efficiency of purification procedures. In addition it also gives the amount of entanglement that has to be used to create a given quantum state.

2. Relative entropy of Entanglement [8, 109, 125]: The relative entropy of entanglement are based on distinguishability and geometrical distance. The idea is to compare a given quantum state σ of a pair of particles with separable states. The relative entropy of entanglement of a given state σ is defined by

$$E_{RE}(\sigma) = \min_{\rho \in M} D(\sigma || \rho) \quad (1.8)$$

Here 'M' denotes the set of all separable states and D can be any function that describes a measure of separation between two density operators. A particular form of the function D is the relative entropy which is defined as $S(\sigma || \rho) = \text{Tr}\{\sigma \ln \sigma - \sigma \ln \rho\}$.

3. Concurrence [7, 109, 143, 148]: Wootters gave out, for the mixed state $\hat{\rho}$ of two qubits, the concurrence is

$$C = \max(\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, 0) \quad (1.9)$$

where the λ_i , in decreasing order, are the square roots of the eigen values of the matrix $\rho^{\frac{1}{2}}(\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)\rho^{\frac{1}{2}}$ with ρ^* denoting the complex conjugation of ρ in the computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ and σ_y denoting the Pauli operator.

The entanglement of formation E_F can then be expressed as a function of C given in

(1.9), namely

$$E_F = -\frac{1 + \sqrt{1 - C^2}}{2} \log_2 \frac{1 + \sqrt{1 - C^2}}{2} - \frac{1 - \sqrt{1 - C^2}}{2} \log_2 \frac{1 - \sqrt{1 - C^2}}{2} \quad (1.10)$$

The concurrence and entanglement of formation satisfy convexity. For the two qubit state given in equation (1.3), the concurrence is given by

$$C = 2\sqrt{\lambda_1 \lambda_2} \quad (1.11)$$

The concurrence in equation (1.11) can also be written as

$$C = \sqrt{2(1 - \text{Tr} \rho_A^2)} \quad (1.12)$$

4. Negativity [7, 109, 143]: The negativity E_N of a mixed state ρ is defined by

$$E_N(\rho) = 2 \sum_j \max(0, -\mu_j) \quad (1.13)$$

where μ_j are the eigenvalues of the partial transpose ρ^Γ of the density matrix ρ of the system.

The negativity [142] of a bipartite system described by the density matrix ρ can also be expressed in the form as

$$E_N(\rho) = \frac{\|\rho^{T_A}\|_1 - 1}{2} \quad (1.14)$$

where ρ^{T_A} is the partial transpose with respect to the subsystem A, and $\|\cdot\|$ denotes the trace norm.

Negativity does not change under LOCC. It measures how negative the eigenvalues of the density matrix are after the partial transpose is taken. For pure states it has been proven that the negativity is exactly equal to the concurrence [141]. For mixed states, Eisert and Plenio [57] conjectured that negativity never exceeds concurrence. K.Audenaert, F.Verstraete, T.D.Bie and B.D.Moor [7] proved the conjecture of Eisert and Plenio that concurrence can never be smaller than negativity. For higher dimension, the negativity can be generalized as [104]

$$E_N(\rho) = \frac{\|\rho^{T_A}\|_1 - 1}{d - 1} \quad (1.15)$$

where d is the smaller of the dimensions of the bipartite system.

1.1.6 Entanglement swapping

Entanglement swapping [17, 158] is a method that enables one to entangle two quantum systems that do not have direct interaction with one another. In order to illustrate entanglement swapping, we first define Bell states as $\phi^\pm \equiv \frac{(|00\rangle \pm |11\rangle)}{\sqrt{2}}$ and $\psi^\pm \equiv \frac{(|01\rangle \pm |10\rangle)}{\sqrt{2}}$. Suppose two distant parties, Alice and Bob, share ϕ_{12}^+ and ϕ_{34}^+ where Alice has qubits '1' and '4', and Bob possesses '2' and '3'. A measurement is performed on qubits '2' and '3' with the Bell basis $\{\phi^\pm, \psi^\pm\}$, then the total state $\phi_{12}^+ \otimes \phi_{34}^+$ is projected onto $|\eta_1\rangle = \phi_{23}^+ \otimes \phi_{14}^+$, $|\eta_2\rangle = \phi_{23}^- \otimes \phi_{14}^-$, $|\eta_3\rangle = \psi_{23}^+ \otimes \psi_{14}^+$, $|\eta_4\rangle = \psi_{23}^- \otimes \psi_{14}^-$ with equal probability of $\frac{1}{4}$ for each. Previous entanglement between qubits '1' and '2', and '3' and '4' are now swapped into entanglement between qubits '2' and '3', and '1' and '4'. Although we considered entanglement swapping with the initial state $\phi_{12}^+ \otimes \phi_{34}^+$, similar results can be achieved with other Bell states.

S.Bose et.al. [17] generalized the procedure of entanglement swapping and obtained a scheme for manipulating entanglement in multiparticle systems. They showed that this scheme can be regarded as a method of generating entangled states of many particles. An explicit scheme that generalizes entanglement swapping to the case of generating a 3-particle GHZ state from three Bell pairs has been presented by Zukowski et.al. [158] The standard entanglement swapping helps to save a significant amount of time when one wants to supply two distant users with a pair of atoms or electrons (or any particle possessing mass) in a Bell state from some central source. The entanglement swapping can be used, with some probability which we quantify, to correct amplitude errors that might develop in maximally entangled states during propagation.

1.2 Distance measure between quantum states

1.2.1 Fidelity

A measure of distance between quantum states is the fidelity [111]. The fidelity of states ρ and σ is defined to be

$$F(\rho, \sigma) \equiv \text{Tr} \sqrt{\rho^{\frac{1}{2}} \sigma \rho^{\frac{1}{2}}} \quad (1.16)$$

When ρ and σ commute the quantum fidelity $F(\rho, \sigma)$ reduces to the classical fidelity.

The fidelity between a pure state $|\psi\rangle$ and an arbitrary state ρ is defined by

$$F(|\psi\rangle, \rho) = \sqrt{\langle\psi|\rho|\psi\rangle} \quad (1.17)$$

Properties of Fidelity:

1. The fidelity is symmetric in its inputs, i.e. $F(\rho, \sigma) = F(\sigma, \rho)$.
2. The fidelity $F(\rho, \sigma)$ lies between 0 and 1 (including 0 and 1), i.e. $0 \leq F(\rho, \sigma) \leq 1$.
 - (i) $F(\rho, \sigma) = 0$, iff ρ and σ have support on orthogonal subspaces.
 - (ii) $F(\rho, \sigma) = 1$, iff $\rho = \sigma$.
3. The fidelity is not a metric but it can be converted into a metric by suitably defining it. If we define the angle between states ρ and σ by $A(\rho, \sigma) \equiv \arccos F(\rho, \sigma)$, then fidelity turn out to be a metric because the angle between two points on the sphere is a metric.
4. Fidelity is monotonic under quantum operation.
5. Fidelity has the strong concavity property.

1.2.2 Hilbert-Schmidt (H-S) distance

The Hilbert-Schmidt distance [66, 115] is defined by

$$D_{HS}(\sigma, \rho) = \|\sigma - \rho\|^2 = \text{Tr}[(\sigma - \rho)^2] \quad (1.18)$$

The Hilbert-Schmidt distance defined in (1.18) satisfies all the criterion of the distance function D which is defined below:

Let S be set of density operators on the Hilbert space H.

Let $D : S \times S \rightarrow R$ be a function satisfying the following conditions:

- D1. $D(\sigma, \rho) \geq 0$ for any $\sigma, \rho \in S$ and the equality holds when $\sigma = \rho$.
- D2. $D(\Theta\sigma, \Theta\rho) \leq D(\sigma, \rho)$ for any $\sigma, \rho \in S$ and for any completely positive trace preserving map Θ on the space of operators on H.

H-S distance is easier to calculate and also it serves as a good measure of quantifying the distance between the pure states. It is conjectured that the Hilbert-Schmidt distance is a reasonable candidate of a distance to generate an entanglement measure [140]. Also it

is shown that the quantum relative entropy and the Bures metric satisfy (D1) and (D2) [139].

Later in this thesis we will use frequently the H-S distance measure.

1.3 *No-Cloning Theorem: A Brief History*

1.3.1 *Wigner's prediction*

In 1961, E.P.Wigner [146], assumed that there be a 'living state' $|\psi\rangle$ which is given in a quantum mechanical sense in a finite dimensional Hilbert space H^N . He then noticed that, among all the possible unitary transformations, those that transform $|\Psi_i\rangle = |\psi\rangle|w\rangle$ to $|\Psi_f\rangle = |\psi\rangle|\psi\rangle|r\rangle$, where $|r\rangle$ is the rejected part of the nutrient state $|w\rangle$, are a negligible set but he failed to notice that no transformation realizes that task for arbitrary $|\psi\rangle$, which would have been the no-cloning theorem. From his observation Wigner concluded that biological reproduction "appears to be a miracle from the point of view of the physicist". But nowadays we know that his description of the living state as a quantum mechanical state is not correct because quantum mechanics would not permit the accurate replication of biological information. On the other hand genetic information encoded in a living state can be safely copied because it is never encoded in superposition states - they would be instantly destroyed by decoherence [159]. Pati [122] had shown that Wigner's replicating machine for a species can be ruled out simply based on the linearity of the quantum theory. Thus there does not exist any replicating machine for a living organism in the quantum mechanical sense.

1.3.2 *Herbert's Argument*

In 1982, Nick Herbert [88] put forward an unconventional proposal "FLASH", where he used quantum correlations [56] to communicate faster than light. The word FLASH is an acronym for "First Light Amplification Superluminal Hookup". His apparatus consisted of idealized laser gain tube which would have macroscopically distinguishable outputs when the input was a single arbitrarily polarized photon. The output of the

apparatus contains noise which is due to the combined effect of 'stimulated emission' and 'spontaneous emission'. In spite of the production of noise his claim was that at least statistically, the noise would not prevent identifying the polarization of the incoming photon. Herbert's argument can also be given in the following way:

Let us consider two distant parties, Alice and Bob, sharing two qubits in the singlet state

$$|\psi^-\rangle_{AB} = \frac{1}{\sqrt{2}}(|01\rangle_{AB} - |10\rangle_{AB}) \quad (1.19)$$

The first qubit A belongs to Alice and the second qubit B belongs to Bob. Now, Alice measures σ_x or σ_z in the basis $\{|+\rangle, |-\rangle\}$ or $\{|0\rangle, |1\rangle\}$ on her qubit, where $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$.

If Alice measures σ_x on her qubit, then

$$|\psi^-\rangle_{AB} \quad \text{transforms to} \quad \frac{-1}{\sqrt{2}}[(\sigma_x|-\rangle)_A \otimes |+\rangle_B + (\sigma_x|+\rangle)_A \otimes |-\rangle_B] \quad (1.20)$$

If Alice measures σ_z on her qubit, then

$$|\psi^-\rangle_{AB} \quad \text{transforms to} \quad \frac{1}{\sqrt{2}}[(\sigma_z|0\rangle)_A \otimes |1\rangle_B + (\sigma_z|1\rangle)_A \otimes |0\rangle_B] \quad (1.21)$$

After Alice's measurement and without any communication with her, Bob finds his qubit in the random mixture. For the first case (1.20), Bob finds his qubit in the mixed state described by the density operator $\frac{1}{2}(|+\rangle\langle+| + |-\rangle\langle-|) = \frac{1}{2}I$ and for the second case (1.21), he finds his qubit in the random mixture $\frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) = \frac{1}{2}I$. Now Bob cannot identify the basis in which Alice perform her measurement for the following two reasons: (i) The mixed state appeared in Bob's place due to Alice's measurement is random in both cases and (ii) We are taking into account the fact that there is no classical communication in between them.

But if Bob uses his perfect cloner to copy his qubit, then according to the measurement performed by Alice, Bob finds his qubit in two different mixed states.

If Alice measures σ_x , then the mixture at Bob's place is given by

$$\begin{aligned}\rho_x &= \frac{1}{2}(|++\rangle\langle++| + |+-\rangle\langle+-|) \\ &= \frac{1}{4}(|00\rangle\langle 00| + |00\rangle\langle 11| + |01\rangle\langle 01| + |01\rangle\langle 10| + |10\rangle\langle 01| + |10\rangle\langle 10| + |11\rangle\langle 00| + \\ &\quad |11\rangle\langle 11|)\end{aligned}\tag{1.22}$$

If Alice measures σ_z , then the mixed state at Bob's place is given by the reduced density operator

$$\rho_z = \frac{1}{2}(|00\rangle\langle 00| + |11\rangle\langle 11|)\tag{1.23}$$

From equations (1.22) and (1.23), it is clear that $\rho_x \neq \rho_z$. Therefore, by applying the perfect cloner, Bob can distinguish the measurement performed by Alice without any classical communication with her.

Putting the argument in this way, Herbert thought that the two distant partners can communicate with each other faster than the speed of light. But his thought experiment was proved to be wrong by Wootters and Zurek.

1.3.3 Wootters and Zurek No-cloning Theorem

In 1982, Wootters and Zurek [147], in their pioneering work proved that "an arbitrary quantum state cannot be cloned". This theorem is popularly known as 'no-cloning theorem'. Therefore, no-cloning theorem ruled out the Herbert's argument on superluminal communication between the two distant partners. The impossibility of cloning of an unknown qubit makes quantum information different from classical information. In quantum regime, no-cloning theorem only prohibits the construction of an apparatus which will amplify arbitrary non-orthogonal states but it does not rule out the possibility of a device which amplifies the orthogonal states. Since orthogonal states can be thought of as a different states of classical information so there is no question of contradiction between the no-cloning theorem and the cloning of classical information. Next we will give the proof of no-cloning theorem using linearity and unitarity of the quantum mechanics.

1. Proof of No-cloning theorem by Linearity

If possible, let there exist a perfect amplifying device which would have the following effect on an incoming arbitrary quantum state $|s\rangle$:

$$|s\rangle|\Sigma\rangle|Q_i\rangle \rightarrow |s\rangle|s\rangle|Q_f\rangle \quad (1.24)$$

where $|Q_i\rangle$ and $|Q_f\rangle$ are the initial and final state of the device respectively. $|\Sigma\rangle$ represents a blank state on which the input state is copied.

Let an arbitrary pure quantum state $|s\rangle$ be given by

$$|s\rangle = \alpha|0\rangle + \beta|1\rangle \quad (1.25)$$

with $\alpha^2 + |\beta|^2 = 1$.

The action of the amplifier on the two orthogonal states $|0\rangle$ and $|1\rangle$ separately is given by

$$|0\rangle|\Sigma\rangle|Q_i\rangle \rightarrow |0\rangle|0\rangle|Q_0\rangle \quad (1.26)$$

$$|1\rangle|\Sigma\rangle|Q_i\rangle \rightarrow |1\rangle|1\rangle|Q_1\rangle \quad (1.27)$$

Now due to the linear structure of the quantum mechanics, the interaction between the amplifying device and the state (1.25) is given by

$$|s\rangle|\Sigma\rangle|Q_i\rangle \rightarrow \alpha|00\rangle|Q_0\rangle + \beta|11\rangle|Q_1\rangle \quad (1.28)$$

Since we assume that the amplifier is perfect, so

$$|s\rangle|\Sigma\rangle|Q_i\rangle \rightarrow |s\rangle|s\rangle|Q_f\rangle = (\alpha^2|00\rangle + \alpha\beta|01\rangle + \beta\alpha|10\rangle + \beta^2|11\rangle)|Q_f\rangle \quad (1.29)$$

Therefore, in general, the equations (1.28) and (1.29) are not identical. Thus, we arrive at a contradiction. Hence, there does not exist any perfect amplifier which could copy an arbitrary quantum state.

The equations (1.28) and (1.29) are identical only when $\alpha = 0$ or $\beta = 0$. This observation tells us that the information stored in the state $|0\rangle$ or $|1\rangle$ can be perfectly copied while the information stored in the arbitrary superposition of the states $|0\rangle$ and $|1\rangle$ cannot be

perfectly copied.

2. Proof of No-cloning theorem by Unitarity

Let us assume that we have a perfect quantum cloning machine which can copy an unknown quantum state $|\psi\rangle$. Suppose $|\Sigma\rangle$ represents a blank state onto which the unknown quantum state is to be copied. Therefore, before interaction with the copying machine, the joint state is given by

$$|\psi\rangle \otimes |\Sigma\rangle \quad (1.30)$$

Let U be the unitary evolution which governs the copying procedure of the perfect quantum cloning machine and its effect on the combined state (1.30) is given by

$$U(|\psi\rangle \otimes |\Sigma\rangle) = |\psi\rangle \otimes |\psi\rangle \quad (1.31)$$

Suppose equation (1.31) is valid for two pure unknown quantum states, $|\xi\rangle$ and $|\eta\rangle$. Then we have

$$U(|\xi\rangle \otimes |\Sigma\rangle) = |\xi\rangle \otimes |\xi\rangle \quad (1.32)$$

$$U(|\eta\rangle \otimes |\Sigma\rangle) = |\eta\rangle \otimes |\eta\rangle \quad (1.33)$$

The inner product of (1.32) and (1.33) gives

$$\begin{aligned} \langle \eta | \xi \rangle &= (\langle \eta | \xi \rangle)^2 \\ \Rightarrow \langle \eta | \xi \rangle (1 - \langle \eta | \xi \rangle) &= 0 \\ \Rightarrow \text{either } \langle \eta | \xi \rangle &= 0 \text{ or } \langle \eta | \xi \rangle = 1 \end{aligned} \quad (1.34)$$

Equation (1.34) implies that the quantum states $|\xi\rangle$ and $|\eta\rangle$ are either orthogonal or identical. Therefore, perfect quantum cloner can be constructed for only orthogonal states $|0\rangle$ and $|1\rangle$.

1.4 Description of quantum cloning machines

1.4.1 State dependent quantum cloning machines

In this section, we study the state dependent quantum cloning machines. When the quality of the copies at the output of the quantum cloner depends on the input state, the machine is said to be state dependent quantum cloning machine [20, 40, 87, 147].

1. Wootters and Zurek quantum cloning machine

In the computational basis states $|0\rangle$ and $|1\rangle$, Wootters and Zurek quantum cloning transformation is given by,

$$|0\rangle_a |\Sigma\rangle_b |Q_i\rangle_x \longrightarrow |0\rangle_a |0\rangle_b |Q_0\rangle_x \quad (1.35)$$

$$|1\rangle_a |\Sigma\rangle_b |Q_i\rangle_x \longrightarrow |1\rangle_a |1\rangle_b |Q_1\rangle_x \quad (1.36)$$

The system labeled by 'a' is the input mode, while the other system 'b' represents the qubit onto which information is copied and is analogous to "blank paper" in a copier. $|Q_i\rangle_x$ is the initial machine state vector and $|Q_0\rangle_x$ and $|Q_1\rangle_x$ are the final machine state vectors.

Unitarity of the transformation gives,

$$\langle Q_i | Q_i \rangle = \langle Q_0 | Q_0 \rangle = \langle Q_1 | Q_1 \rangle = 1 \quad (1.37)$$

Let us now consider pure superposition state given by,

$$|\chi\rangle_a = \alpha|0\rangle_a + \beta|1\rangle_a \quad (1.38)$$

For simplicity we will assume the probability amplitudes to be real and $\alpha^2 + \beta^2 = 1$.

The density matrix of the state $|\chi\rangle$ in the input mode 'a' is given by,

$$\rho_a^{id} = |\chi\rangle\langle\chi| = \alpha^2|0\rangle\langle 0| + \alpha\beta|0\rangle\langle 1| + \alpha\beta|1\rangle\langle 0| + \beta^2|1\rangle\langle 1| \quad (1.39)$$

After applying the cloning transformation (1.35-1.36) the arbitrary quantum state $|\chi\rangle$ given in equation (1.38) takes the form

$$|\chi^{out}\rangle_{abx} = \alpha|0\rangle_a |0\rangle_b |Q_0\rangle_x + \beta|1\rangle_a |1\rangle_b |Q_1\rangle_x \quad (1.40)$$

If it is assumed that two copying machine states $|Q_0\rangle$ and $|Q_1\rangle$ are orthonormal then the reduced density operator $\rho_{ab}^{(out)}$ is given by

$$\rho_{ab}^{(out)} = Tr_x[\rho_{abx}^{(out)}] = \alpha^2|00\rangle\langle 00| + \beta^2|11\rangle\langle 11| \quad (1.41)$$

where $\rho_{abx}^{(out)} = |\chi^{out}\rangle_{abx}\langle\chi^{out}|$.

The reduced density operators describing the original and the copy mode are given by,

$$\rho_a^{(out)} = Tr_b[\rho_{ab}^{(out)}] = \alpha^2|0\rangle\langle 0| + \beta^2|1\rangle\langle 1| \quad (1.42)$$

$$\rho_b^{(out)} = Tr_a[\rho_{ab}^{(out)}] = \alpha^2|0\rangle\langle 0| + \beta^2|1\rangle\langle 1| \quad (1.43)$$

The copying quality i.e. the distance between the density matrix of the input state $\rho_a^{(id)}$ and the reduced density matrix $\rho_a^{(out)}$ ($\rho_b^{(out)}$) of the output state can be measured by Hilbert-Schmidt norm given by

$$D_a = Tr[\rho_a^{(id)} - \rho_a^{(out)}]^2 \quad (1.44)$$

In this case,

$$D_a = 2\alpha^2\beta^2 = 2\alpha^2(1 - \alpha^2) \quad (1.45)$$

D_a is called copy quality index.

Wootters and Zurek (W-Z) quantum cloning machine can be regarded as a state- dependent quantum cloning machine because the distance between the pure states depends on the input state parameter α . Thus, for some values of α , the distance is minuscule while for some values of α , the distance is very large. Hence Wootters and Zurek (W-Z) quantum cloning machine works perfectly for some inputs and badly for some others. Since D_a depends on α^2 , the average distortion is calculated over all input states, i.e., over all α^2 lying between 0 and 1, which is

$$\overline{D_a} = \int_0^1 D_a(\alpha^2) d\alpha^2 = \frac{1}{3} \quad (1.46)$$

(a) The copy quality indices and the entanglement indices:

Buzek and Hillery [28] expresses the entanglement indices D_{ab}^1 , D_{ab}^2 , D_{ab}^3 in terms of the copy quality indices D_a and D_b . D_{ab}^1 expresses the "H-S distance" between the actual two mode density operator ρ_{ab}^{out} and a tensor product of density operators ρ_a^{out} and ρ_b^{out} . D_{ab}^2 measures the "H-S distance" between density operator ρ_{ab}^{out} and a tensor product of ρ_a^{id} and ρ_b^{id} . D_{ab}^3 represents the "H-S distance" between the tensor product of density operators ρ_a^{id} and ρ_b^{id} and a tensor product of ρ_a^{out} and ρ_b^{out} .

Also we note that the two outputs produced by the Wootters and Zurek quantum cloning machine are same and thus $D_a = D_b = 2\alpha^2\beta^2$ (equ. 1.45). The entanglement indices can be expressed in terms of D_a or D_b or both in the following way:

$$\begin{aligned}
D_{ab}^1 &= Tr[\rho_{ab}^{(out)} - \rho_a^{(out)} \otimes \rho_b^{(out)}]^2 \\
&= (2\alpha^2\beta^2).(2\alpha^2\beta^2) \\
&= D_a.D_b \\
&= D_a^2 = D_b^2
\end{aligned} \tag{1.47}$$

$$\begin{aligned}
D_{ab}^2 &= Tr[\rho_{ab}^{(out)} - \rho_a^{(id)} \otimes \rho_b^{(id)}]^2 \\
&= 8\alpha^4\beta^4 + 4\alpha^2\beta^2(\alpha^4 + \beta^4) \\
&= 2\alpha^2\beta^2 + 2\alpha^2\beta^2 \\
&= D_a + D_b \\
&= 2D_a = 2D_b
\end{aligned} \tag{1.48}$$

$$\begin{aligned}
D_{ab}^3 &= Tr[\rho_a^{(id)} \otimes \rho_a^{(id)} - \rho_a^{(out)} \otimes \rho_b^{(out)}]^2 \\
&= 4\alpha^4\beta^4 + 4\alpha^2\beta^2(\alpha^4 + \beta^4) \\
&= 2\alpha^2\beta^2 + 2\alpha^2\beta^2 - (2\alpha^2\beta^2).(2\alpha^2\beta^2) \\
&= D_a + D_b - D_a.D_b \\
&= D_a(2 - D_a) = D_b(2 - D_b)
\end{aligned} \tag{1.49}$$

(b) Wootters and Zurek quantum cloning machine in higher dimension:

In two dimensional Hilbert space the relationships between the copying quality indices and the entanglement indices are established by Buzek and Hillery. So it is natural to ask a question whether those relationships between the copying quality indices and the entanglement indices depend on the dimension of the state space? The answer is given in the affirmative by M.Ying [149]. He studied the Wootters's and Zurek quantum copying machine on a higher dimensional state space and established the inequalities which described the relationship among the copying quality indices D_a and D_b and the entanglement indices $D_{ab}^1, D_{ab}^2, D_{ab}^3$.

The transformation rule for Wootters and Zurek quantum copying machine in 'n' dimension can be defined by

$$|k\rangle_a |\Sigma\rangle_b |Q_i\rangle_x \rightarrow |k\rangle_a |k\rangle_b |Q_k\rangle_x \quad (k = 0, 1, \dots, n-1) \quad (1.50)$$

The unitarity of the transformation gives

$$\langle Q_i | Q_i \rangle = \langle Q_k | Q_k \rangle = 1 \quad (k = 0, 1, \dots, n-1) \quad (1.51)$$

Further, it is assumed that the machine state vectors are mutually orthogonal, i.e. $\langle Q_k | Q_l \rangle = 0$ for $0 \leq k, j \leq n-1$ and $k \neq j$.

The n-dimensional pure state is given by

$$|\chi\rangle_a = \sum_{k=0}^{n-1} \alpha_k |k\rangle_a \quad (1.52)$$

with $\sum \alpha_k^2 = 1$.

For simplicity it is assumed that the probability amplitudes α_k ($k=0,1,\dots,n-1$) are all real numbers.

The density operator for the generalised n-dimensional input state is

$$\rho_a^{(id)} = \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} \alpha_k \alpha_j |k\rangle \langle j| \quad (1.53)$$

The action of the cloning machine on the n-dimensional input state is given by

$$|\chi\rangle_a |\Sigma\rangle_b |Q_i\rangle \rightarrow |\psi\rangle_{abx}^{out} \equiv \sum_{k=0}^{n-1} \alpha_k |k\rangle_a |k\rangle_b |Q_k\rangle_x \quad (1.54)$$

After tracing out the machine state vector in mode 'x', the reduced density operator describing the output state is given by

$$\rho_{ab}^{(out)} = Tr_x[\rho_{abx}^{(out)}] = \sum_{k=0}^{n-1} \alpha_k^2 |kk\rangle\langle kk| \quad (1.55)$$

Furthermore, the reduced density operator describing the state in mode 'a' and 'b' is given by

$$\rho_a^{(out)} = Tr_b[\rho_{ab}^{(out)}] = \sum_{k=0}^{n-1} \alpha_k^2 |k\rangle\langle k| \quad (1.56)$$

and

$$\rho_b^{(out)} = Tr_a[\rho_{ab}^{(out)}] = \sum_{k=0}^{n-1} \alpha_k^2 |k\rangle\langle k| \quad (1.57)$$

The copying quality indices are given by

$$D_a(n) = Tr[\rho_a^{(id)} - \rho_a^{(out)}]^2, \quad D_b(n) = Tr[\rho_b^{(id)} - \rho_b^{(out)}]^2$$

and the entanglement indices are given by

$$D_{ab}^1(n) = Tr[\rho_{ab}^{(out)} - \rho_a^{(out)} \otimes \rho_b^{(out)}]^2, \quad D_{ab}^2(n) = Tr[\rho_{ab}^{(out)} - \rho_{ab}^{(id)}]^2,$$

$$D_{ab}^3(n) = Tr[\rho_{ab}^{(id)} - \rho_a^{(out)} \otimes \rho_b^{(out)}]^2,$$

$$\text{where } \rho_{ab}^{(id)} = \rho_a^{(id)} \otimes \rho_b^{(id)}, \quad \rho_a^{(id)} = \rho_b^{(id)}, \quad \rho_a^{(out)} = \rho_b^{(out)}.$$

The relations between the entanglement indices and the copying quality indices are the following [149]:

First Inequality:

$$D_a(n).D_b(n) - \frac{(n-1)(n-2)}{n^2} \leq D_{ab}^1(n) \leq D_a(n).D_b(n) \quad (1.58)$$

The minimum value of $D_{ab}^1(n) - D_a(n).D_b(n)$ is attained when $\alpha_0^2 = \alpha_1^2 = \dots = \alpha_{n-1}^2 = \frac{1}{n}$. The maximum value of $D_{ab}^1(n) - D_a(n).D_b(n)$ is attained at each of the points $(\alpha_0^2, \alpha_1^2, \dots, \alpha_{n-1}^2) \equiv (1, 0, \dots, 0), (0, 1, 0, \dots, 0), \dots, (0, 0, \dots, 1)$.

Second Inequality:

$$[D_a(n) + D_b(n)] - \frac{(n-1)(n-2)}{n^2} \leq D_{ab}^2(n) \leq D_a(n) + D_b(n) \quad (1.59)$$

The minimum value of $D_{ab}^2(n) - [D_a(n) + D_b(n)]$ is attained when $\alpha_0^2 = \alpha_1^2 = \dots = \alpha_{n-1}^2 = \frac{1}{n}$. The maximum value of $D_{ab}^2(n) - [D_a(n) + D_b(n)]$ is attained at each of the

points $(\alpha_0^2, \alpha_1^2, \dots, \alpha_{n-1}^2) \equiv (1, 0, \dots, 0), (0, 1, 0, \dots, 0), \dots, (0, 0, \dots, 1)$.

Third Inequality:

$$D_a(n) + D_b(n) - D_{ab}^1(n) - \frac{(n-1)(n-2)}{n^2} \leq D_{ab}^3(n) \leq D_a(n) + D_b(n) - D_{ab}^1(n) \quad (1.60)$$

The minimum value of $D_{ab}^3(n) - [D_a(n) + D_b(n) - D_{ab}^1(n)]$ is attained when $\alpha_0^2 = \alpha_1^2 = \dots = \alpha_{n-1}^2 = \frac{1}{n}$. The maximum value of $D_{ab}^3(n) - [D_a(n) + D_b(n) - D_{ab}^1(n)]$ is attained at each of the points $(\alpha_0^2, \alpha_1^2, \dots, \alpha_{n-1}^2) \equiv (1, 0, \dots, 0), (0, 1, 0, \dots, 0), \dots, (0, 0, \dots, 1)$.

2. Bruss, DiVincenzo, Ekert, Fuchs, Macchiavello and Smolin's quantum cloning machine

The deterministic state dependent quantum cloner was first studied by Bruss, DiVincenzo, Ekert, Fuchs, Macchiavello and Smolin [20]. They designed a optimal state dependent cloner which copy the qubit selected from an ensemble containing only two equiprobable non-orthogonal quantum states $|a\rangle$ and $|b\rangle$. They have also shown that a priori knowledge about the input state makes the performance of the quantum cloning machine better than the universal quantum cloning machine in the sense of higher fidelity.

The optimal fidelity as obtained by Bruss et.al. [20] for state dependent cloner is given by the formula:

$$F = \frac{1}{2} + \frac{\sqrt{2}}{32S}(1+S)(3-3S+\sqrt{9S^2-2S+1}) \times \sqrt{3S^2+2S-1+(1-S)\sqrt{9S^2-2S+1}} \quad (1.61)$$

where $S = |\langle a|b \rangle|$.

The minimum value of F is approximately equal to 0.987 and it is achieved for $S = \frac{1}{2}$. In quantum cryptography [11, 13, 55, 135], the eavesdropper's main concern is not in copying the quantum information encoded in the two non-orthogonal quantum states but rather in optimizing the tradeoff between the classical information made available to her versus the disturbance inflicted upon the original qubit. In this respect, optimal

state dependent quantum cloner played a crucial role. Eavesdropper can use the state dependent quantum cloning machine to copy the original qubit in transit between a sender and a receiver. Then she resent the original qubit to the receiver. Thereafter, she can obtain some information from the cloned qubit by measuring it. Also since the disturbance upon the original qubit is very low due to the application of state dependent cloner, eavesdropper can steal the information in the midway without giving any clue to sender and receiver.

If one could get a success to construct a nearly perfect cloner then it will certainly solve some problem like state estimation problem [21, 106, 154] but sidewise it will create a problem in quantum cryptography. In quantum cryptography, the third party Eve could steal information using quantum cloning machine which is nearly perfect. So the optimal quantum cloning machine not only helps us but also it has some negative effects like it also helps the information hacker 'Eve'. We observe that for this job state dependent cloner would be the more efficient candidate than state independent cloner.

1.4.2 *State independent (Universal) quantum cloning machine*

In this section, we study the state independent quantum cloning machines. When the quality of the two identical copies at the output are independent of the input state, the machine is said to be state independent or universal [20, 28, 31, 59, 68, 79].

1. Buzek and Hillery quantum cloning machine ($1 \rightarrow 2$ type)

Wootters and Zurek considered a quantum copying machine (1.35-1.36) which demonstrated that if it copies two basis vectors perfectly then it cannot copy the superpositions of these vectors perfectly, i.e. there does not exist a quantum copier which can copy arbitrary qubit without introducing errors. Even though ideal copying is prohibited by the laws of quantum mechanics for an arbitrary state, it is still possible to design quantum copiers that operate reasonably well. The first universal quantum cloning machine was constructed by Buzek and Hillery (in 1996) [28].

In particular, the universal quantum cloning machine is specified by the following conditions [31]:

(i) The state of the original system and its quantum copy at the output of the quantum cloner, described by the density operators ρ_a^{out} and ρ_b^{out} , respectively are identical, i.e.,

$$\rho_a^{out} = \rho_b^{out} \quad (1.62)$$

(ii) If no a priori information about the input state of the original system is available, then to fulfil the requirement of equal copy quality of all pure states, one should design a quantum copier in such a way that the distances between the density operators of each system at the output ρ_j^{out} where $j = a, b$ and the ideal density operator ρ_j^{id} which describes the input state of the original mode are input state independent. Quantitatively this means that if we employ the square of the Hilbert-Schmidt norm

$$d(\hat{\rho}_1; \hat{\rho}_2) = Tr[(\hat{\rho}_1 - \hat{\rho}_2)^2] \quad (1.63)$$

as a measure of distance between two operators, then the quantum copier should be such that

$$d(\hat{\rho}_j^{out}; \hat{\rho}_j^{id}) = constant, \quad j = a, b \quad (1.64)$$

Here we note that other measures of the distance between two density operators such as Bures distance and trace norm can also be used to specify the universal cloning transformation [101]. The final form of the transformation does not depend on the choice of the measure.

(iii) Finally, to make the quality of the copies better, it is required that the copies are as close as possible to the input state.

To construct the universal quantum cloning machine which obeys the above three conditions, Buzek and Hillery proposed a cloning transformation given below:

$$|0\rangle_a |\Sigma\rangle_b |Q\rangle_x \longrightarrow |0\rangle_a |0\rangle_b |Q_0\rangle_x + [|0\rangle_a |1\rangle_b + |1\rangle_a |0\rangle_b] |Y_0\rangle_x \quad (1.65)$$

$$|1\rangle_a |\Sigma\rangle_b |Q\rangle_x \longrightarrow |1\rangle_a |1\rangle_b |Q_1\rangle_x + [|0\rangle_a |1\rangle_b + |1\rangle_a |0\rangle_b] |Y_1\rangle_x \quad (1.66)$$

The unitarity of the transformation gives

$$\langle Q_i | Q_i \rangle + 2\langle Y_i | Y_i \rangle = 1, \quad i = 0, 1 \quad (1.67)$$

$$\langle Y_0 | Y_1 \rangle = \langle Y_1 | Y_0 \rangle = 0 \quad (1.68)$$

Further, it is assumed that

$$\langle Q_i | Y_i \rangle = 0, \quad i = 0, 1 \quad (1.69)$$

$$\langle Q_0 | Q_1 \rangle = 0 \quad (1.70)$$

The quantum cloning machine (1.65-1.66) copies the input state (1.38) and produces two-qubit output described by the density operator

$$\begin{aligned} \rho_{ab}^{(out)} = & \alpha^2 |00\rangle \langle 00| \langle Q_0 | Q_0 \rangle + \sqrt{2}\alpha\beta |00\rangle \langle + | \langle Y_1 | Q_0 \rangle + \sqrt{2}\alpha\beta | + \rangle \langle 00| \langle Q_0 | Y_1 \rangle \\ & + [2\alpha^2 \langle Y_0 | Y_0 \rangle + 2\beta^2 \langle Y_1 | Y_1 \rangle] | + \rangle \langle + | + \sqrt{2}\alpha\beta | + \rangle \langle 11| \langle Q_1 | Y_0 \rangle \\ & + \sqrt{2}\alpha\beta |11\rangle \langle + | \langle Y_0 | Q_1 \rangle + \beta^2 |11\rangle \langle 11| \langle Q_1 | Q_1 \rangle \end{aligned} \quad (1.71)$$

where $|+\rangle = \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle)$.

The reduced density operator describing the original mode can be obtained by taking partial trace over the copy mode and it reads as

$$\rho_a^{(out)} = [\alpha^2 + \xi(\beta^2 - \alpha^2)] |0\rangle \langle 0| + \alpha\beta\eta(|0\rangle \langle 1| + |1\rangle \langle 0|) + [\beta^2 + \xi(\beta^2 - \alpha^2)] |1\rangle \langle 1| \quad (1.72)$$

where $\langle Y_0 | Y_0 \rangle = \langle Y_1 | Y_1 \rangle = \xi$, $\langle Y_0 | Q_1 \rangle = \langle Q_0 | Y_1 \rangle = \langle Q_1 | Y_0 \rangle = \langle Y_1 | Q_0 \rangle = \frac{\eta}{2}$,

ξ and η are two free parameters and they can be determined from condition (ii) mentioned above and the criteria that the distance between the two-qubit output density matrix $\rho_{ab}^{(out)}$ and the ideal two-qubit output $\rho_{ab}^{(id)}$ be input state independent.

The density operator $\rho_b^{(out)}$ describing the copy mode is exactly same as the density operator $\rho_a^{(out)}$ describing the original mode.

Now the Hilbert Schmidt norm for the density operators $\rho_a^{(id)}$ and $\rho_a^{(out)}$ is given by,

$$D_a = 2\xi^2(4\alpha^4 - 4\alpha^2 + 1) + 2\alpha^2\beta^2(\eta - 1)^2 \quad (1.73)$$

with $0 \leq \xi \leq \frac{1}{2}$ and $0 \leq \eta \leq 2\sqrt{\xi(1-2\xi)} \leq \frac{1}{\sqrt{2}}$ which follows from Schwarz inequality.

Now to satisfy the condition (ii) of universal quantum cloning machine described above, the Hilbert Schmidt norm D_a must be independent of the parameter α^2 .

$$\frac{\partial D_a}{\partial \alpha^2} = 0 \implies \eta = 1 - 2\xi \quad (1.74)$$

Using the relation $\eta = 1 - 2\xi$, the equation (1.73) reduces to

$$D_a = 2\xi^2 \quad (1.75)$$

The value of the parameter ξ can be determined from the condition that the distance between two-qubit density operators $\rho_{ab}^{(id)}$ and $\rho_{ab}^{(out)}$ be input state independent, i.e.

$$\frac{\partial D_{ab}^2}{\partial \alpha^2} = 0 \quad (1.76)$$

where $D_{ab}^2 = \text{Tr}[\rho_{ab}^{(out)} - \rho_{ab}^{(id)}]^2 = 8\xi^2 + 2\alpha^2(1 - \alpha^2)(1 - 6\xi)$.

Solving equation (1.76), we find $\xi = \frac{1}{6}$. For this value of ξ , the norm D_{ab}^2 is independent of α^2 and its value is equal to $\frac{2}{9}$.

Also for $\xi = \frac{1}{6}$, the distance between the single qubit state at the output of the copying machine and the input state is given by

$$D_a = \frac{1}{18} \quad (1.77)$$

Next we will show that the two-qubit density operator ρ_{ab}^{out} given by the equation (1.71) is inseparable.

In the case of two qubits, we can utilize the necessary and sufficient condition [91, 123] which states that the partial transpose of a 4×4 density matrix

$$W_4 = \begin{vmatrix} \rho_{00,00} & \rho_{01,00} & \rho_{00,10} & \rho_{01,10} \\ \rho_{00,01} & \rho_{01,01} & \rho_{00,11} & \rho_{01,11} \\ \rho_{10,00} & \rho_{11,00} & \rho_{10,10} & \rho_{11,10} \\ \rho_{10,01} & \rho_{11,01} & \rho_{10,11} & \rho_{11,11} \end{vmatrix} \quad \text{be negative for non-separability to hold.}$$

Using the value of the parameter $\xi = \frac{1}{6}$, the equation (1.71) can be rewritten as

$$\begin{aligned} \rho_{ab}^{out} = & \frac{2\alpha^2}{3} |00\rangle\langle 00| + \frac{\alpha\beta}{3} (|00\rangle\langle 01| + |00\rangle\langle 10| + |01\rangle\langle 00| + |10\rangle\langle 00|) + \frac{\alpha\beta}{3} (|01\rangle\langle 11| \\ & + |10\rangle\langle 11| + |11\rangle\langle 01| + |11\rangle\langle 10|) + \frac{1}{6} (|01\rangle\langle 01| + |01\rangle\langle 10| + |10\rangle\langle 01| \\ & + |10\rangle\langle 10|) + \frac{2\beta^2}{3} |11\rangle\langle 11| \end{aligned} \quad (1.78)$$

The partial transpose of a 4×4 density matrix is given by

$$W_4 = \begin{vmatrix} \frac{2\alpha^2}{3} & \frac{\alpha\beta}{3} & \frac{\alpha\beta}{3} & \frac{1}{6} \\ \frac{\alpha\beta}{3} & \frac{1}{6} & 0 & \frac{\alpha\beta}{3} \\ \frac{\alpha\beta}{3} & 0 & \frac{1}{6} & \frac{\alpha\beta}{3} \\ \frac{1}{6} & \frac{\alpha\beta}{3} & \frac{\alpha\beta}{3} & \frac{2\beta^2}{3} \end{vmatrix} = -\frac{1}{6^4} \quad (1.79)$$

Equation (1.79) implies that W_4 is negative for all values of α^2 , therefore the two qubit at the output of the quantum copier is inseparable for any arbitrary input state.

Furthermore, Gisin and Massar [79] showed that the universal quantum cloning machine satisfies criteria (iii) i.e. the universal quantum cloning transformation defined by equations (1.65-1.66) is optimal in the sense that it produces best quality copies at the output.

Therefore the optimal unitary transformation which implements the universal quantum cloning machine is given by

$$|0\rangle_a |\rangle_b |Q\rangle_x \longrightarrow \sqrt{\frac{2}{3}} |00\rangle_{ab} |\uparrow\rangle_x + \sqrt{\frac{1}{3}} |+\rangle_{ab} |\downarrow\rangle_x \quad (1.80)$$

$$|1\rangle_a |\rangle_b |Q\rangle_x \longrightarrow \sqrt{\frac{2}{3}} |11\rangle_{ab} |\downarrow\rangle_x + \sqrt{\frac{1}{3}} |+\rangle_{ab} |\uparrow\rangle_x \quad (1.81)$$

The state space of the cloning machine is two dimensional and it is spanned by the orthogonal vectors $|\uparrow\rangle_x$ and $|\downarrow\rangle_x$.

The reduced density operators describing the state of both copies at the output of the universal quantum cloner (1.80-1.81) are equal and they can be expressed as

$$\rho_a^{\hat{o}ut} = \rho_b^{\hat{o}ut} = \frac{5}{6} |\psi\rangle_a \langle\psi| + \frac{1}{6} |\psi_\perp\rangle_a \langle\psi_\perp| \quad (1.82)$$

where

$$|\psi\rangle_a = \alpha |0\rangle_a + \beta |1\rangle_a, \quad |\psi_\perp\rangle_a = \beta^* |0\rangle_a - \alpha^* |1\rangle_a \quad (1.83)$$

From equation (1.79), it is clear that the mixed state at the output of the cloner is entangled. Bruss and Macchiavello [26] studied the entanglement properties of the output state of a universal quantum cloning machine. Without any loss of generality, they studied the entanglement structure of the cloning output for an input basis state $|0\rangle$ given in equation (1.80). Their investigations are about the amount of entanglement between the two clones and between an ancilla and a clone.

The reduced density matrices ρ_{ab} for two clones and ρ_{ax} for one clone and ancilla are

given by

$$\rho_{ab} = \begin{vmatrix} \frac{2}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{6} & \frac{1}{6} & 0 \\ 0 & \frac{1}{6} & \frac{1}{6} & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} \quad \text{and} \quad \rho_{ax} = \begin{vmatrix} \frac{2}{3} & 0 & 0 & \frac{1}{3} \\ 0 & \frac{1}{6} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{1}{6} \end{vmatrix} \quad (1.84)$$

In the case of the cloning output, the concurrences defined in equation (1.9), for the mixed states described by density matrices ρ_{ab} and ρ_{ax} , are calculated to be $C_{ab} = \frac{1}{3}$ and $C_{ax} = \frac{2}{3}$. Further using the relation (1.10), the entanglement of formation for the density matrices ρ_{ab} and ρ_{ax} are found out to be $E_{F,ab} \simeq 0.1873$ and $E_{F,ax} \simeq 0.55$ respectively. Therefore, we can observe that the entanglement between clone and ancilla is higher than between the two copies. Hence, after the cloning procedure, the information is distributed in such a way that some of it are in the copies, some are in the entanglement between the copies, some are in the copy machine, and some are in the entanglement between the copies and the copy machine. The information in the entanglement and in the copy machine is effectively lost.

Almost simultaneously with Gisin and Massar [79] but independently Bruss, DiVincenzo, Ekert, Fuchs, Machiavello and Smolin [20] have constructed the class of unitary transformations for the optimal universal symmetric quantum cloner.

Here the class of unitary cloning transformation is given by

$$U|0\rangle_a|\Sigma\rangle_b|Q\rangle_x = \sqrt{\frac{2}{3}} e^{i\delta_a}|00\rangle_{ab}|Q_0\rangle_x + \sqrt{\frac{1}{6}} e^{i\delta_{\bar{a}}}(|01\rangle_{ab} + |10\rangle_{ab})|Q_1\rangle_x \quad (1.85)$$

$$U|1\rangle_a|\Sigma\rangle_b|Q\rangle_x = \sqrt{\frac{2}{3}} e^{i\delta_{\bar{a}}}|11\rangle_{ab}|Q_1\rangle_x + \sqrt{\frac{1}{6}} e^{i\delta_a}(|01\rangle_{ab} + |10\rangle_{ab})|Q_0\rangle_x \quad (1.86)$$

where $\langle Q_0|Q_1\rangle = 0$ and $\delta_a, \delta_{\bar{a}}$ denotes the phase factors.

Each output state of the quantum cloning machine is input state independent if and only if the reduced density operator takes the form $\rho^{(out)} = \eta|\psi\rangle^{(in)}\langle\psi| + \frac{1}{2}(1-\eta)I$. Therefore, the quality of the clones is defined by the fidelity $F = {}^{(in)}\langle\psi|\rho^{(out)}|\psi\rangle^{(in)} = \frac{1}{2}(1+\eta)$, where η denotes the reduction factor.

Bruss et.al. [20] found that the quantum cloning transformation (1.85-1.86) would be optimal if $\eta = \frac{2}{3}$.

The corresponding optimal cloning fidelity is

$$F_{opt} = \frac{5}{6} \quad (1.87)$$

Also we note that if $\delta_a = \delta_{\bar{a}} = 0$ then it reduces to Buzek-Hillery universal quantum cloning transformation.

2. Universal symmetric cloner ($1 \rightarrow M$ type and $N \rightarrow M$ type)

In 1997, Gisin and Massar [79] introduced the idea of generating M identical copies from one qubit input. They also extended their idea for an arbitrary number $N(< M)$ of input qubits.

The $1 \rightarrow M$ quantum cloning machine, when acting on an arbitrary input state $|\psi\rangle$, is described by the following unitary operator:

$$U_{1 \rightarrow M} |\psi\rangle \otimes R = \sum_{j=0}^{M-1} \alpha_j |(M-j)\psi, j\psi^\perp\rangle \otimes R_j(\psi) \quad (1.88)$$

where $\alpha_j = \sqrt{\frac{2(M-j)}{M(M+1)}}$ and $R_j(\psi)$ represents the internal state of the quantum cloning machine with $\langle R_j(\psi) | R_k(\psi) \rangle = 0$ for $j \neq k$.

The density matrix $\rho^{(out)} = F_{1,M} |\psi\rangle\langle\psi| + (1 - F_{1,M}) |\psi^\perp\rangle\langle\psi^\perp|$ describing the output qubits is the same for all copies, where the fidelity $F_{1,M}$ is given by

$$\begin{aligned} F_{1,M} &= \sum_{j=0}^{M-1} \text{Prob}(j \text{ errors in the } (M-1) \text{ last qubits}) \\ &= \frac{2M+1}{3M}. \end{aligned} \quad (1.89)$$

A more general quantum cloning machine that takes N identical qubits all prepared in the state $|\psi\rangle$ into $M(> N)$ identical copies is described by

$$U_{N \rightarrow M} |\psi\rangle^{\otimes N} \otimes R = \sum_{j=0}^{M-N} \alpha_j |(M-j)\psi, j\psi^\perp\rangle \otimes R_j(\psi) \quad (1.90)$$

where $\alpha_j = \sqrt{\frac{N+1}{M+1}} \sqrt{\frac{(M-N)!(M-j)!}{(M-N-j)!M!}}$.

The fidelity of each output qubit of the more generalized quantum cloning machine is given by

$$F_{N,M} = \frac{M(N+1) + N}{M(N+2)}. \quad (1.91)$$

For pure input states the fidelity given in equation (1.91) which is achieved by the cloning transformation (1.90) was shown to be optimal in [19]. The fidelity $F_{N,M}$ tends to $\frac{N+1}{N+2}$ as $M \rightarrow \infty$, which is the optimal fidelity achievable by carrying out a measurement on N identical input qubits.

Moreover, Gisin and Massar discussed some special cases of the generalized quantum cloning machine defined in equation (1.90).

Case-1: If $N=1$ and $M=2$, then $N \rightarrow M$ quantum cloning machine reduces to $1 \rightarrow 2$ Buzek-Hillery universal quantum cloning machine. The fidelity of each output qubit is $\frac{5}{6}$.

Case-2: If $N=1$ and $M=M_1(> 1)$ but finite, then $N \rightarrow M$ quantum cloning machine reduces to $1 \rightarrow M_1$ quantum cloning machine. The fidelity of each output qubit is $F_{1,M_1} = \frac{2M_1+1}{3M_1}$. Now if we assume that the quantum cloning machine could produce infinite number of copies i.e. if $M_1 \rightarrow \infty$ then $F_{1,\infty} \rightarrow \frac{2}{3}$. In this case the cloning fidelity is equal to the fidelity of measurement i.e. the overlapping between the states before and after measurement for a given single unknown quantum state. Further, we note that when the number of clones M_1 grows for fixed N , the cloning fidelity decreases.

Case-3: If $N=N_1$ and $M=N_1 + 1$ both are finite, then $N \rightarrow M$ quantum cloning machine reduces to $N_1 \rightarrow N_1 + 1$ quantum cloning machine. In this case, The fidelity of each output qubit reduces to $F_{N_1,N_1+1} = \frac{N_1^2+3N_1+1}{N_1^2+3N_1+2}$ which tends to 1 as $N_1 \rightarrow \infty$.

3. Universal symmetric quantum cloner in d -Dimension [31, 152]

In 1998, Buzek and Hillery [31] proposed a universal cloning transformation of states in a d -dimensional Hilbert space.

The cloning transformation in a d -dimensional Hilbert space is given by

$$\begin{aligned} |\Psi_i\rangle_a |\Sigma\rangle_b |Q\rangle_x \rightarrow & \sqrt{\frac{2}{d+1}} |\Psi_i\rangle_a |\Psi_i\rangle_b |Q_i\rangle_x + \sqrt{\frac{1}{2(d+1)}} \sum_{j \neq i}^d (|\Psi_i\rangle_a |\Psi_j\rangle_b \\ & + |\Psi_j\rangle_a |\Psi_i\rangle_b) |Q_j\rangle_x \end{aligned} \quad (1.92)$$

The density operator describing each copy at the output can be written in the scaled form as

$$\rho_j^{(out)} = \eta \rho_j^{(id)} + \frac{1-\eta}{d} \hat{I} \quad (1.93)$$

where $\rho_j^{(id)} = |\psi\rangle\langle\psi|$ is the density operator describing the input state which is going to be cloned and η is called reduction factor or scaling factor which is given by

$$\eta = \frac{d+2}{2(d+1)} \quad (1.94)$$

As $d \rightarrow \infty$, the scaling factor $\eta \rightarrow \frac{1}{2}$.

The fidelity of the copies in terms of the reduction factor is given by

$$F(d) = \frac{\eta(d-1) + 1}{d} \quad (1.95)$$

For 2-dimensional case, the scaling factor and the fidelity takes the value $\eta = \frac{2}{3}$ and $F(2) = \frac{5}{6}$ which was shown to be optimal value by Gisin and Massar and also by Bruss et.al. Further we note that if we consider the cloning transformation of states in an infinite dimensional Hilbert space then one can copy the quantum information with at most fidelity $\frac{1}{2}$. That means in an infinite dimensional Hilbert space, we cannot extract much information about an arbitrary quantum state using quantum cloning machine. The reason behind the poor copying of quantum information contained in a higher dimensional state space can be explained by the von Neumann entropy. The von Neumann entropy measures the degree of entanglement between the copies and the copier and is given by

$$S = \ln(d+1) - \frac{2 \ln 2}{d+1} \quad (1.96)$$

It is clear from equation (1.96) that the entropy does not depend on the input state to be copied but it depends on the dimension of the state space. Also the entropic function is an increasing function of the dimension d . Therefore, with the increase of the dimension d , the entanglement between the copies and the copier also increases. Probably this is the reason why cloning machine fails to produce better quality copies in higher

dimension.

Fan, Matsumoto and Wadati [59] constructed an optimal N to M ($N < M$) quantum cloning transformation for d -dimensional quantum system. Their proposed cloning transformation is given by

$$U_{N \rightarrow M} |\mathbf{n}\rangle \otimes R = \sum_{j=0}^{M-N} \alpha_{\mathbf{n}, \mathbf{j}} |\mathbf{n} + \mathbf{j}\rangle \otimes R_{\mathbf{j}} \quad (1.97)$$

where $|\mathbf{n}\rangle = |n_1, \dots, n_d\rangle$ is a completely symmetric and normalized state, $R_{\mathbf{j}}$ denotes the orthogonal normalized internal states of the universal quantum cloning machine,

$\sum_{k=1}^d j_k = M - N$ and

$$\alpha_{\mathbf{n}, \mathbf{j}} = \sqrt{\frac{(M-N)!(N+d-1)!}{(M+d-1)!}} \sqrt{\prod_{k=1}^d \frac{(n_k + j_k)!}{n_k! j_k!}} \quad (1.98)$$

The state of each d -level clone is described by the reduced density operator

$$\rho^{out} = \eta_{N,M}(d) |\psi\rangle\langle\psi| + \frac{(1 - \eta_{N,M}(d))}{d} \hat{I} \quad (1.99)$$

where $\eta_{N,M}(d) = \frac{N(M+d)}{M(N+d)}$ denotes the scaling factor for qudits.

The fidelity of the copy qudit produced from the quantum cloning machine (1.97) is given by

$$F_{N,M}(d) = \frac{N(d-1) + M(N+1)}{(d+N)M} \quad (1.100)$$

The fidelity $F_{N,M}(d)$ was shown to be optimal by Werner [145] and Keyl and Werner [99].

Note: (i) For $d=2$, the fidelity $F_{N,M}(2)$ reduces to the fidelity $F_{N,M}$ of the generalised $N \rightarrow M$ quantum cloning machine for qubit.

(ii) When $N=1$ and $M=2$, the performance of the quantum cloning machine for qudit is given in terms of the fidelity $F_{1,2}(d) = \frac{d+3}{2(d+1)}$.

(iii) In the limit $d \rightarrow \infty$, the fidelity $F_{N,M}(\infty)$ tends to $\frac{N}{M}$. Further, if we want to produce large number of copies of finite number of input in higher dimensional systems i.e. if $N \ll M$, then the existing quantum cloning machine produces poor quality copies.

(iv) For sufficiently large M and finite N and d , the fidelity $F_{N,\infty}(d)$ tends to $\frac{N+1}{N+d}$ which

is the optimal fidelity for state estimation of N copies of a d -dimensional quantum system. For $d=2$, the cloning fidelity tends to the optimal measurement fidelity $\frac{N+1}{N+2}$ for N identical qubits. This expression for measurement fidelity originally derived by Massar and Popescu [108].

1.4.3 Probabilistic cloning

One could design a quantum cloning machine which will copy only states from a particular set of allowed input states. This type of cloning machine produces better quality copies than the universal quantum cloning machine. In fact, it is possible for the copier to be perfect for certain small size of the input sets. If the input set contains any two non-orthogonal states, then it is impossible to build a perfect quantum copier [89]. Also it is not possible to build a perfect copier for input sets of more than two states. Therefore, the quantum copier which produces perfect copies and works every time does not exist in nature. But if we relax the latter condition i.e. if we allow the quantum cloning machine to fail to produce perfect copies for sometime, then such type of quantum cloning machine exists in nature and they are called probabilistic quantum cloning machines [6, 50, 69, 128, 138, 153, 155, 156]. Probabilistic quantum cloning machine performs measurements and unitary operations, with a post selection of the measurement results and hence the desired copies are produced only with certain probabilities. Also we cannot exclude the fact that there are some probability for which the cloning machine fails to produce the perfect copies and in those cases the copies would be discarded.

In 1997, Duan and Guo [50, 52] first proposed such type of quantum cloning machine which produces with some probability perfect copies of two non-orthogonal states. They showed that two non-orthogonal quantum states secretly chosen from a certain set $S = \{|\Psi_0\rangle, |\Psi_1\rangle\}$ can be perfectly cloned with some probability less than unity. Few months later, they generalised their result and showed that non orthogonal quantum states secretly chosen from a certain set $S = \{|\Psi_1\rangle, |\Psi_2\rangle, \dots, |\Psi_n\rangle\}$ can be cloned probabilistically by a unitary evolution together with a reduction process.

Theorem 1.1: The n non-orthogonal states $|\Psi_1\rangle, |\Psi_2\rangle, \dots, |\Psi_n\rangle$ can be probabilis-

tically cloned with unit fidelity by the same cloning machine if and only if they are linearly-independent.

C-W Zhang, Z-Y Wang, C-F Li and G.C.Guo [153] have considered the realizations of quantum probabilistic identifying and cloning machines by physical means. They showed that the unitary representation and the Hamiltonian of probabilistic cloning and identifying machines are determined by the probabilities of successes. The logic networks are obtained by decomposing the unitary representation into universal quantum logic operations. They also discussed the robustness of the networks and found that if error occurs in the input target system, it can be detected and the to-be-cloned states can be recycled.

C-W Zhang, C-F Li, Z-Y Wang and G.C.Guo [156] proposed a probabilistic quantum cloning scheme using Greenberger-Horne-Zeilinger (GHZ) states [85], Bell basis measurements, single-qubit unitary operations and generalized measurements. The single-qubit generalized measurement is performed by the unitary transformation on the composite system of that qubit and the auxiliary probe with reduction measurement of the probe. They showed that their scheme may be used in experiment to clone the states of one particle to those of two different particles with higher probability and less GHZ resources. J.Fiurasek [69] have investigated the optimal probabilistic realizations of several important quantum information processing tasks such as the optimal cloning of quantum states and purification of mixed quantum states. The performance of these probabilistic operations is quantified by the average fidelity between the ideal (generally mixed) states ρ^{in} and actual output pure states $|\psi^{out}\rangle$. He derived a simple formula for the maximum achievable average fidelity and provided an explicit prescription how to construct a trace-decreasing completely positive map that reaches the maximum average fidelity F_{max} given by

$$F_{max} = \max[\text{eig}(A^{-1}R)] \quad (1.101)$$

Where, $A = \int_{S_{in}} (\rho_{in}^T \otimes I_{out}) d\rho_{in}$ and $R = \int_{S_{in}} (\rho_{in}^T \otimes \psi_{out}) d\rho_{in}$. Further, it was shown that the fidelity of probabilistic cloning can be strictly higher than the maximal fidelity

of deterministic cloning even if the set of the cloned states is linearly dependent and continuous. This improvement in fidelity is achieved at the expense of a certain fraction of unsuccessful events when the probabilistic transformation fails and does not produce any output state.

K.Azuma, J.Shimamura, M.Koashi and N.Imoto [6] studied the probabilistic cloning of a mutually non-orthogonal set of pure states $\{|\psi_1\rangle, |\psi_2\rangle\}$, with the help of supplementary information in the form of pure states $\{|\phi_1\rangle, |\phi_2\rangle\}$. They showed that the best efficiency of producing m copies is always achieved by a two-step protocol in which the helping party first attempts to produce $m-1$ copies from the supplementary state, and if it fails, then the original state is used to produce m copies. To perform the two-step protocol, two types of probabilistic cloning machines are used:

- (i) Original state $\{|\psi_i\rangle\}_{i=1,2}$ is copied by the machine $\{|\psi_i\rangle \longrightarrow^{\gamma_i^A} |\psi_i\rangle^{\otimes m}\}_{i=1,2}$ with probability $\gamma_1^A = \gamma_2^A = \frac{1-\langle\psi_1|\psi_2\rangle}{1-|\langle\psi_1|\psi_2\rangle|^m}$
- (ii) Supplementary information in terms of pure state $\{|\phi_i\rangle\}_{i=1,2}$ is copied by the machine $\{|\phi_i\rangle \longrightarrow^{\gamma_i^B} |\psi_i\rangle^{\otimes m-1}\}_{i=1,2}$ with probability $\gamma_1^B = \gamma_2^B = \frac{1-|\langle\phi_1|\phi_2\rangle|}{1-|\langle\psi_1|\psi_2\rangle|^{m-1}}$.

Therefore, using these machines in the two-step protocol, an overall success probability γ_{totmax} is given by

$$\gamma_{totmax} = \gamma_1^B + (1 - \gamma_1^B)\gamma_1^A = \frac{1 - |\langle\psi_1|\psi_2\rangle\langle\phi_1|\phi_2\rangle|}{1 - |\langle\psi_1|\psi_2\rangle|^m} \quad (1.102)$$

It was further shown that when the number of states exceeds two, the best efficiency is not always achieved by such a protocol.

1.4.4 Phase covariant quantum cloning

Bruss, Cinchetti, Ariano, and Macchiavello [24] were the first who studied the cloning transformations for a restricted set of pure input states of the form

$$|\psi_\phi\rangle = \frac{1}{\sqrt{2}}[|0\rangle + e^{i\phi}|1\rangle] \quad (1.103)$$

where the parameter ϕ represents the angle between the Bloch vector and the x-axis and $\phi \in [0, 2\pi)$. The qubits of this form are called equatorial qubits because the z-component

of their Bloch vector is zero. The importance of equatorial qubits lies in the fact that the quantum cryptographic experiments require the equatorial states rather than the states that span the whole Bloch sphere. The cloning transformations that can clone arbitrary equatorial qubits are called phase covariant quantum cloners [24, 42, 43, 58, 61, 98] because they keep the quality of the copies same for all equatorial qubits or in other words the fidelity does not depend on the parameter ϕ .

Bruss, Cinchetti, Ariano, and Macchiavello [24] proposed the following 1 to 2 cloning transformation for the input (1.103),

$$|0\rangle_a |\Sigma\rangle_b |Q\rangle_x \rightarrow [(\frac{1}{2} + \sqrt{\frac{1}{8}})|00\rangle_{ab} + (\frac{1}{2} - \sqrt{\frac{1}{8}})|11\rangle_{ab}] |\uparrow\rangle_x + \frac{1}{2} |+\rangle_{ab} |\downarrow\rangle_x \quad (1.104)$$

$$|1\rangle_a |\Sigma\rangle_b |Q\rangle_x \rightarrow [(\frac{1}{2} + \sqrt{\frac{1}{8}})|11\rangle_{ab} + (\frac{1}{2} - \sqrt{\frac{1}{8}})|00\rangle_{ab}] |\downarrow\rangle_x + \frac{1}{2} |+\rangle_{ab} |\uparrow\rangle_x \quad (1.105)$$

where $|\uparrow\rangle_x$ and $|\downarrow\rangle_x$ denote the orthogonal machine state vectors and

$$|+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle).$$

The reduced density operator of both copies at the output can be expressed as

$$\rho^{out} = (\frac{1}{2} + \sqrt{\frac{1}{8}})|\psi_\phi\rangle\langle\psi_\phi| + (\frac{1}{2} - \sqrt{\frac{1}{8}})|\psi_{\phi,\perp}\rangle\langle\psi_{\phi,\perp}| \quad (1.106)$$

where the state $|\psi_{\phi,\perp}\rangle$ is orthogonal to the state $|\psi_\phi\rangle$.

The optimal fidelity of 1 to 2 phase covariant cloning transformation is given by

$$F_{1,2}^{phase} = \frac{1}{2} + \sqrt{\frac{1}{8}}. \quad (1.107)$$

We note that the fidelity of the phase covariant quantum cloning machine is greater than the fidelity of the universal quantum cloning machine. This is due to the fact that more information about the input qubit is given to the phase covariant quantum cloning machine. For x - y equatorial qubits, Fan, Matsumoto, Wang, Wadati's [60] conjecture was that the general N to M ($M > N$) quantum cloning transformation would be

(i) when $M - N$ is even,

$$U_{N \rightarrow M} |(N-j) \uparrow, j \downarrow\rangle \otimes R = |(\frac{M+N-2j}{2}) \uparrow, (\frac{M-N+2j}{2}) \downarrow\rangle \otimes R_L \quad (1.108)$$

(ii) when $M - N$ is odd,

$$U_{N \rightarrow M} |(N-j) \uparrow, j \downarrow\rangle \otimes R = \frac{1}{\sqrt{2}} |(\frac{M+N-2j+1}{2}) \uparrow, (\frac{M-N+2j-1}{2}) \downarrow\rangle \otimes R_L \\ + \frac{1}{\sqrt{2}} |(\frac{M+N-j-1}{2}) \uparrow, (\frac{M-N+2j+1}{2}) \downarrow\rangle \otimes R_{L+1} \quad (1.109)$$

The corresponding fidelities for the above two cases are given by

(i) when $M - N$ is even,

$$F_{N,M}^{phase} = \frac{1}{2} + \frac{1}{2^N} \sum_{j=0}^{N-1} \frac{N!}{j!(N-j-1)!} \times \sqrt{\frac{(M-N+2j+2)(M+N-2j)}{4M^2(j+1)(N-j)}} \quad (1.110)$$

(ii) when $M - N$ is odd,

$$F_{N,M}^{phase} = \frac{1}{2} + \frac{1}{2^{N+1}} \sum_{j=0}^{N-1} \frac{N!}{j!(N-j-1)!} \times \frac{1}{\sqrt{4M^2(j+1)(N-j)}} \times \\ (\sqrt{(M-N+2j+1)(M+N-2j+1)} \\ + \sqrt{(M-N+2j+3)(M+N-2j-1)}) \quad (1.111)$$

Note:

1. When $N=1$ and $M > 1$,

$$F_{1,M}^{phase} = \frac{1}{2} + \frac{\sqrt{M(M+2)}}{4M} \quad \text{when } M \text{ is even} \\ = \frac{1}{2} + \frac{(M+1)}{4M} \quad \text{when } M \text{ is odd} \quad (1.112)$$

2. In particular, when $N=1$ and $M=2$, the fidelity of each output qubit is given by $F_{1,2}^{phase} = \frac{1}{2} + \sqrt{\frac{1}{8}}$. Moreover, Fan, Matsumoto, Wang, Wadati [60] showed that the copied qubits are separable for the case of optimal phase-covariant quantum cloning. This observation makes the equatorial states unique in the sense that they are the only states which give rise to separable density matrix for the output copies.

3. When $N=1$ and $M=3$, the fidelity of 1 to 3 phase covariant quantum cloning machine is $\frac{5}{6}$ which is equal to the fidelity of 1 to 2 universal Buzek-Hillery quantum cloning machine.

4. If we make $M \rightarrow \infty$ in equations (1.110) and (1.111), then the fidelity in the limiting

sense is given by

$$F_{N,\infty}^{phase} = \frac{1}{2} + \frac{1}{2^{N+1}} \sum_{j=0}^{N-1} \frac{N!}{j!(N-j-1)!} \times \sqrt{\frac{1}{(j+1)(N-j)}} \quad (1.113)$$

The conjecture [60] about N to M phase covariant quantum cloning transformation seems to be correct because the optimal fidelity of $N \rightarrow \infty$ quantum cloning equals to the corresponding fidelity for optimal covariant quantum phase estimation of equatorial qubits originally derived by Derka, Buzek and Ekert [49].

Further, Karimipour and Rezakhani [98] investigated the phase covariant quantum cloning of the states on the Bloch sphere which have a definite z component of spin. They showed that it is always possible to clone a spin state $|\mathbf{n}\rangle$ with a fidelity higher than the universal value and that of equatorial states, if the third component of its spin $\langle \mathbf{n} | \sigma_z | \mathbf{n} \rangle$ is known.

Now to clone a general two level state $|\mathbf{n}\rangle = \cos \frac{\theta}{2} |0\rangle + e^{i\phi} \sin \frac{\theta}{2} |1\rangle$, they considered the following cloning transformation

$$U|0\rangle_a |\Sigma\rangle_b |Q\rangle_x = \nu |00\rangle_{ab} |\uparrow\rangle_x + \mu (|01\rangle_{ab} + |10\rangle_{ab}) |\downarrow\rangle_x \quad (1.114)$$

$$U|1\rangle_a |\Sigma\rangle_b |Q\rangle_x = \nu |11\rangle_{ab} |\downarrow\rangle_x + \mu (|01\rangle_{ab} + |10\rangle_{ab}) |\uparrow\rangle_x \quad (1.115)$$

where $|\uparrow\rangle, |\downarrow\rangle$ denotes the orthogonal machine state vectors.

When the state $|\mathbf{n}\rangle$ is acted upon by the cloning machine (1.114-1.115), each copy at the output is described by the reduced density operator

$$\rho^{out} = \mu^2 I + 2\mu\nu |\mathbf{n}\rangle \langle \mathbf{n}| + (\nu^2 - 2\mu\nu) \cos^2 \frac{\theta}{2} |0\rangle \langle 0| + \sin^2 \frac{\theta}{2} |1\rangle \langle 1| \quad (1.116)$$

where I denotes the identity operator in 2-dimensional Hilbert space.

The fidelity of the cloning is given by

$$F(\theta) = \frac{1}{2} + \mu \sqrt{1 - 2\mu^2} + \left(\frac{1 - 2\mu^2}{2} - \mu \sqrt{1 - 2\mu^2} \right) \langle \sigma_z \rangle^2 \quad (1.117)$$

where $\langle \sigma_z \rangle \equiv \langle \mathbf{n} | \sigma_z | \mathbf{n} \rangle = \cos \theta$

Also

$$D_{ab}^1(\theta) = K_1 \cos^8 \frac{\theta}{2} + K_2 \cos^6 \frac{\theta}{2} + K_3 \cos^4 \frac{\theta}{2} + K_4 \cos^2 \frac{\theta}{2} + K_5 \quad (1.118)$$

Where $K_1 = 576\mu^8 - 768\mu^6 + 352\mu^4 - 64\mu^2 + 4$, $K_2 = -1152\mu^8 + 1536\mu^6 - 704\mu^4 + 128\mu^2 - 8$, $K_3 = 672\mu^8 - 928\mu^6 + 424\mu^4 - 72\mu^2 + 4$, $K_4 = -96\mu^8 + 160\mu^6 - 72\mu^4 + 8\mu^2$, $K_5 = 4\mu^8 + 2\mu^4$

$$D_{ab}^2(\theta) = 8\mu^4 - (6\mu^4 + \mu^2 + 2\mu\nu - 1)\sin^2\theta \quad (1.119)$$

Now for some fixed value of θ , the fidelity $F(\theta)$ attains its optimal value when

$$\mu^2 = \frac{1}{4} \left(1 - \frac{1}{\sqrt{1 + 2\tan^4\theta}} \right) \quad (1.120)$$

It can be observed that the fidelity of cloning spin states with a definite component of spin along the z direction is higher than the fidelity of cloning spin states with zero third component of its spin. Also the distances D_{ab}^1 and D_{ab}^2 is minimized when μ^2 takes the form given in equation (1.120).

Now we discuss here some other interesting results which one can get from equations (1.117-1.119).

Result-1: If $\frac{1-2\mu^2}{2} - \mu\sqrt{1-2\mu^2} = 0$ then $\mu = \frac{1}{\sqrt{6}}$. Therefore, μ does not depend on the parameter θ and hence the fidelity F and the distances D_{ab}^1 and D_{ab}^2 are also independent of θ . Thus the cloning machine (1.114-1.115) reduces to Buzek-Hillery universal quantum cloning machine. The values of the fidelity and the distances are given by $F = \frac{5}{6}$, $D_{ab}^1 = \frac{19}{324}$ and $D_{ab}^2 = \frac{2}{9}$.

Result-2: If $\langle\sigma_z\rangle = 0$, then $\theta = \frac{\pi}{2}$. $F(\frac{\pi}{2}) = \frac{1}{2} + \mu\sqrt{1-2\mu^2}$. $F(\frac{\pi}{2})$ attains its maximum value when $\mu = \frac{1}{2}$. In this case, the cloning machine (1.114-1.115) reduces to optimal phase covariant quantum cloning machine. Therefore, the optimal value of the fidelity and the distances are given by $F^{opt}(\frac{\pi}{2}) = \frac{1}{2} + \frac{1}{\sqrt{8}}$, $D_{ab}^1(\frac{\pi}{2}) = \frac{9}{64}$, $D_{ab}^2(\frac{\pi}{2}) = \frac{7}{8} - \frac{1}{\sqrt{2}}$.

Fan, Imai, Matsumoto and Wang [61] studied the phase-covariant quantum cloning machine for d-level quantum systems. The optimal 1 to 2 phase-covariant quantum cloning transformation for input state $|\Psi\rangle^{(in)} = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} e^{i\phi_j} |j\rangle\rangle$ is given by

$$U|j\rangle_a|\Sigma\rangle_b|Q\rangle_x = \alpha|j\rangle_a|j\rangle_b|Q_j\rangle_x + \frac{\beta}{\sqrt{2(d-1)}} \sum_{k \neq j}^{d-1} (|j\rangle_a|k\rangle_b + |k\rangle_a|j\rangle_b) |Q_k\rangle_x \quad (1.121)$$

where $\alpha^2 = \frac{1}{2} - \frac{d-2}{2\sqrt{d^2+4d-4}}$, $\beta^2 = \frac{1}{2} + \frac{d-2}{2\sqrt{d^2+4d-4}}$ and $|Q_j\rangle_x$ denote the orthonormal machine states.

The reduced density matrix for a single qudit at the output of the cloning machine is given by

$$\rho^{(out)} = \frac{1}{d} \sum_{j=0}^{d-1} |j\rangle\langle j| + \left(\frac{\alpha\beta}{d} \sqrt{\frac{2}{d-1}} + \frac{\beta^2(d-2)}{2d(d-1)} \right) \sum_{j \neq k} e^{i(\phi_j - \phi_k)} |j\rangle\langle k| \quad (1.122)$$

The optimal fidelity of 1 to 2 phase-covariant quantum cloning machine is given by

$$F_{1,2}^{phase}(d) = \frac{1}{d} + \frac{1}{4d}(d-2 + \sqrt{d^2 + 4d - 4}) \quad (1.123)$$

Note:

1. If we consider the two-level quantum system, then the phase-covariant quantum cloning transformation (1.121) produces two copies of the equatorial qubit with optimal fidelity $\frac{1}{2} + \sqrt{\frac{1}{8}}$ which agree with the previous result given in equation (1.107).
2. In case of qutrits i.e. for 3-dimensional quantum system, the optimal fidelity of phase-covariant quantum cloning machine is found to be $F_{1,2}^{phase}(3) = \frac{5+\sqrt{17}}{12}$. The same result was also obtained by D'Ariano, Presti [46]; Cerf, Durt, Gisin [38]; and Karimipour, Rezakhani [98]. Karimipour and Rezakhani [98] also studied the phase covariant quantum cloning of d-level system that lies on the Bloch sphere with a definite z component of spin.

1.4.5 *Economical quantum cloning*

Until now we have discussed those quantum cloning transformations (cloning machines) in which an additional system called ancilla or machine state is present. But the presence of ancilla significantly affects the recent NMR experiments that were realized for the implementation of cloning operations. The negative effects occurs due to sensitiveness of the ancilla towards decoherence and as a result the achieved cloning fidelity reduces. It was difficult to avoid such effects because the cloning network contained at least ten single-qubit gates and five two-qubit gates. To overcome this problem, it is necessary to construct a cloning network in which less number of quantum gates are required and also it should keep the fidelity of cloning at its optimal level. Fortunately, Niu and Griffiths [113] designed a 2-qubit cloning network in which no external ancilla is required and

that it requires only two single-qubit gates and one two-qubit gate and it requires to control the entanglement of a pair of qubits only. It is thus likely to be quite less noisy than its 3 qubit counter part. The cloning procedure which does not require an extra ancilla or machine state is termed as economical quantum cloning [27, 39, 54, 113] .

Durt and Du [53] analyzed the possibility to reduce 3 qubit one-to-two phase-covariant quantum cloning [71] to 2 qubit economic quantum cloning [113]. They derived a necessary and sufficient condition to characterize the reducibility of 3 qubit cloners to 2 qubit cloners. They showed that when this condition is fulfilled, economic cloning is possible. Durt, Fiurasek and Cerf [54] proved that universal 1 to 2 cloning transformation for any dimension $d \geq 2$ is not possible to implement in an economic way i.e. without an ancilla, just by applying a two-qudit unitary transformation to the original state and a blank copy. They also showed that the 2-dimensional optimal phase-covariant cloner can be realized economically while an economical phase-covariant cloner cannot be constructed for any dimension $d > 2$. The impossibility of the construction of phase-covariant quantum cloner without an ancilla led them to think about the approximate economical phase-covariant quantum cloning machine. The optimal economical phase-covariant cloning transformation which is invariant with respect to the swapping of the two clones and is also phase covariant is given by

$$\begin{aligned} U|k\rangle_a|l\rangle_b &= |k\rangle_a|k\rangle_b, \quad k = l \\ &= \frac{1}{\sqrt{2}}(|k\rangle_a|l\rangle_b + |l\rangle_a|k\rangle_b), \quad k \neq l \end{aligned} \quad (1.124)$$

where $l, k \in \{1, 2, \dots, d-1, d\}$ and initially the blank state is prepared in the state $|l\rangle_b$.

The corresponding fidelity of economical cloning is given by

$$F_{1,2 \text{ opt}}^{\text{econ.phase}}(d) = \frac{1}{2d^2}[d-1 + (d-1 + \sqrt{2})^2] \quad (1.125)$$

Remark:

1. $F_{1,2 \text{ opt}}^{\text{econ.phase}} = F_{1,2 \text{ opt}}^{\text{phase}} = \frac{1}{2} + \sqrt{\frac{1}{8}}$, when $d = 2$. In case of qubit, cheaper (or economical) phase-covariant quantum cloning machine can be constructed with the

same fidelity as optimal phase-covariant quantum cloning machine.

2. $F_{1,2 \text{ opt}}^{\text{econ.phase}} < F_{1,2 \text{ opt}}^{\text{phase}}$, when $d > 2$. In higher dimensional case, approximate phase-covariant quantum cloning machine without ancilla can be designed at the cost of lower fidelity of cloning than the optimal phase-covariant quantum cloning machine with ancilla.

3. $F_{1,2 \text{ opt}}^{\text{econ.phase}}, F_{1,2 \text{ opt}}^{\text{phase}} \rightarrow \frac{1}{2}$ as $d \rightarrow \infty$. That means in case of infinite dimensional Hilbert space neither approximate economical phase-covariant quantum cloning machine nor optimal phase-covariant quantum cloning machine with ancilla play a significant role in cloning procedure because the fidelity is very low in this case.

1.4.6 Asymmetric quantum cloning

N.J.Cerf [35] brought in a new concept of quantum cloning machine which copies the information of a quantum system into two non-identical (approximate) clones. To implement his idea, he introduced a family of Pauli quantum cloning machines that produce two approximate non-identical copies and later he generalized the Pauli quantum cloning machine to any arbitrary dimension and constructed a family of asymmetric Heisenberg quantum cloning machine. J.Fiurasek, R.Filip, and N.J.Cerf [70] investigated asymmetric universal cloning in arbitrary dimension. They proved the optimality of the universal asymmetric $1 \rightarrow 2$ cloning machines and then extended the idea of asymmetric cloning to quantum triplicators, which produce three clones with different fidelity. S.Iblisdir, A.Acin, N.J.Cerf, R.Filip, J.Fiurasek and N.Gisin [94] ; S.Iblisdir, A.Acin and N.Gisin [95] investigated the optimal distribution of quantum information over multipartite systems by introducing the optimal asymmetric $N \rightarrow M_A + M_B$ cloning machine. The cloning machine takes N identical pure input states and produces two sets of clones M_A and M_B with fidelities F_A and F_B respectively. They also analyzed the trade-off between these fidelities. The latter group also generalized the above asymmetric quantum cloning transformation to more than two sets of clones. The experimental implementations of an optimal asymmetric $1 \rightarrow 2$ quantum cloning of a polarization state of photon is recently proposed by R.Filip [67]. The cloning transformation $N \rightarrow M_A + M_B$ has been proven to

be a useful tool when (i) studying the security of some quantum key distribution scheme and (ii) studying the estimation of state by keeping finite number of clones in one set and infinite number (in the limiting sense) of clones in another set.

L.-P.Lamoureux, and N.J.Cerf [102] constructed the class of optimal $1 \rightarrow 2$ phase-covariant quantum cloning machines in any dimension and then extended the concept to the class of asymmetric quantum cloning machines. They studied the balance between the fidelity of two clones and concluded that the relative fidelity between two clones decreases with the dimension.

The following notions are used in the discussion:

Definition-1.1: The cloning machines which produce two approximate non-identical clones are known as asymmetric cloning machines.

Definition-1.2: A Pauli channel is defined using a group of four error operators, (the three Pauli matrices $\sigma_x, \sigma_y, \sigma_z$ and I) which act on state $|\psi\rangle$ by either rotating it by one of the Pauli matrices or leaving it unchanged.

Definition-1.3: Asymmetric cloning machines that produce two output qubits, each emerging from a Pauli channel, are called asymmetric Pauli cloning machines.

1. Asymmetric Pauli cloning machines

If the input qubit X of the Pauli channel is initially in a fully entangled state with a reference qubit R that is unchanged while X is processed by the channel, i.e. if the joint state of input qubit X and the reference qubit is $|\psi\rangle_{RX} = |\phi^+\rangle$, then the joint state of the reference qubit R and the output Y is a mixture of the four Bell states $|\Phi^\pm\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$ and $|\Psi^\pm\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)$ and can be written in the form

$$\rho_{RY} = (1-p)|\Phi^+\rangle\langle\Phi^+| + p_z|\Phi^-\rangle\langle\Phi^-| + p_x|\Psi^+\rangle\langle\Psi^+| + p_y|\Psi^-\rangle\langle\Psi^-| \quad (1.126)$$

where p_x, p_y, p_z denote the probabilities with which a qubit undergoes a phase-flip (σ_z), a bit-flip (σ_x), or their combination ($\sigma_x\sigma_z = -i\sigma_y$) in a Pauli channel and $p = p_x + p_y + p_z$. Instead of defining a Pauli cloning machine by a particular unitary transformation, Cerf [36, 37] characterized a Pauli cloning machine by the four qubit wave function $|\psi\rangle_{RABC}$.

Thus the family of Pauli cloning machines relies on a parametrization of 4-qubit wave functions for which all qubit pairs are in a mixture of Bell states. After cloning, the four qubits R,A,B,C are in a pure state for which ρ_{RA} and ρ_{RB} are mixtures of Bell states (i.e. A and B emerge from a Pauli channel). Also it is assumed that ρ_{RC} to be a Bell mixture.

The four qubit wave function $|\psi\rangle_{RABC}$ for the bipartite partition RA versus BC can be written as a superposition of double Bell states

$$|\psi\rangle_{RA;BC} = \{v|\Phi^+\rangle|\Phi^+\rangle + z|\Phi^-\rangle|\Phi^-\rangle + x|\Psi^+\rangle|\Psi^+\rangle + y|\Psi^-\rangle|\Psi^-\rangle\}_{RA;BC} \quad (1.127)$$

where x,y,z,v are complex amplitudes satisfying the condition $|x|^2 + |y|^2 + |z|^2 + |v|^2 = 1$. For simplicity other possible permutations of the Bell states are not considered here. The first output A emerges from a Pauli channel with probabilities $p_x = |x|^2, p_y = |y|^2$ and $p_z = |z|^2$.

An interesting property of these double Bell states is that they transform into superpositions of double Bell states for the two remaining partitions of the four qubits RABC into two other pairs (RB versus AC) and (RC versus AB).

Therefore, the four qubit wave function $|\psi\rangle_{RABC}$ can also be written for the partition RB versus AC as

$$|\psi\rangle_{RB;AC} = \{v'|\Phi^+\rangle|\Phi^+\rangle + z'|\Phi^-\rangle|\Phi^-\rangle + x'|\Psi^+\rangle|\Psi^+\rangle + y'|\Psi^-\rangle|\Psi^-\rangle\}_{RB;AC} \quad (1.128)$$

with

$$\begin{aligned} v' &= \frac{1}{2}(v + z + x + y) \\ z' &= \frac{1}{2}(v + z - x - y) \\ x' &= \frac{1}{2}(v - z + x - y) \\ y' &= \frac{1}{2}(v - z - x + y) \end{aligned} \quad (1.129)$$

Equation (1.129) implies that when tracing over half of the system, the second output B emerges from a Pauli channel with probabilities $q_x = |x'|^2, q_y = |y'|^2$ and $q_z = |z'|^2$.

The partition RC versus AB

$$|\psi\rangle_{RC;AB} = \{v''|\Phi^+\rangle|\Phi^+\rangle + z''|\Phi^-\rangle|\Phi^-\rangle + x''|\Psi^+\rangle|\Psi^+\rangle + y''|\Psi^-\rangle|\Psi^-\rangle\}_{RC;AB} \quad (1.130)$$

describe the third output C which emerges from a Pauli channel with probabilities $|x''|^2$, $|y''|^2$, $|z''|^2$ and $|v''|^2$ and these probabilities are related with $|x|^2$, $|y|^2$, $|z|^2$ and $|v|^2$ in the following way:

$$\begin{aligned} v'' &= \frac{1}{2}(v + z + x - y) \\ z'' &= \frac{1}{2}(v + z - x + y) \\ x'' &= \frac{1}{2}(v - z + x - y) \\ y'' &= \frac{1}{2}(v - z - x - y) \end{aligned} \quad (1.131)$$

Now the trade-off between the quality of the two copies produced by an asymmetric Pauli cloning machine can be studied by writing the wave function of the whole system of four particles RABC in the following way:

$$|\psi\rangle_{RABC} = \sum_{m,n=0}^1 \alpha_{m,n} |\Phi_{m,n}\rangle_{RA} |\Phi_{m,-n}\rangle_{BC} = \sum_{m,n=0}^1 \beta_{m,n} |\Phi_{m,n}\rangle_{RB} |\Phi_{m,-n}\rangle_{AC} \quad (1.132)$$

where $|\Phi_{m,n}\rangle$ denotes the Bell basis.

The relation between the coefficients $\alpha_{m,n}$ and $\beta_{m,n}$ is given by

$$\beta_{m,n} = \left(\frac{1}{2}\right) \sum_{x,y=0}^1 e^{i\pi(nx-my)} \alpha_{x,y} \quad (1.133)$$

This shows that if one output copy (say, A) is close to perfect, then second output copy B is close to imperfect (i.e. very noisy) and vice versa.

2. Asymmetric Heisenberg cloning machines

Cerf [37] generalized the asymmetric Pauli cloning machine to systems of arbitrary dimensions d and defined a family of asymmetric cloning machines that produces two imperfect copies of the state of an N -dimensional quantum system that emerge from non-identical Heisenberg channels.

The corresponding family of asymmetric quantum cloning machines are called asymmetric Heisenberg cloning machines. A Heisenberg channel is characterized by the N^2 -dimensional probability distribution \mathbf{p} of error operators which a quantum state undergoes in the channel.

The generalized Bell basis is defined by

$$|\Phi_{m,n}\rangle = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} \exp\left(\frac{2\pi i k n}{d}\right) |k\rangle |(k+m) \text{ modulo } d\rangle \quad (1.134)$$

Let us suppose that the input state we wish to clone $|\psi\rangle_A$ is prepared in the maximally entangled state $|\Phi_{0,0}\rangle$, given by (1.134), with a reference state R. The cloning machine is described by a unitary operation U acting on a four qubit state, namely the initial state and another two d-level systems initially prepared in the state $|0\rangle_B$ and $|0\rangle_C$:

$$\begin{aligned} U|\Phi_{0,0}\rangle_{RA}|00\rangle_{BC} = |\psi\rangle_{RABC} &= \sum_{m,n=0}^{d-1} \alpha_{m,n} |\Phi_{m,n}\rangle_{RA} |\Phi_{m,-n}\rangle_{BC} \\ &= \sum_{m,n=0}^{d-1} \beta_{m,n} |\Phi_{m,n}\rangle_{RB} |\Phi_{m,-n}\rangle_{AC} \end{aligned} \quad (1.135)$$

where A, B denotes the two clones, C is the ancilla (cloning machine), $|\Phi_{m,n}\rangle$ is the generalized Bell state and

$$\beta_{m,n} = \left(\frac{1}{d}\right) \sum_{x,y=0}^{d-1} \exp\left(\frac{2\pi i (nx - my)}{d}\right) \alpha_{x,y} \quad (1.136)$$

The action of a Heisenberg cloning machine on an arbitrary input state is given by

$$U|\psi\rangle_A|00\rangle_{BC} = |\chi\rangle_{ABC} = \sum_{m,n=0}^{d-1} \alpha_{m,n} U_{m,n} |\psi\rangle_A |\Phi_{m,-n}\rangle_{BC} \quad (1.137)$$

where $\sum_{m,n=0}^{d-1} |\alpha_{m,n}|^2 = 1$ and $U_{m,n}$ are the error operators which define the Heisenberg group:

$$U_{m,n} = \sum_{k=0}^{d-1} \exp\left(\frac{2\pi i k n}{d}\right) |(k+m) \text{ modulo } d\rangle \langle k| \quad (1.138)$$

We note that for two dimensional case, $U_{m,n}$ become the Pauli operators.

After cloning, the two output copies are described by the density operators

$$\rho_A = \text{Tr}_{BC}(|\chi\rangle_{ABC}\langle\chi|) = \sum_{m,n=0}^{d-1} |\alpha_{m,n}|^2 |\psi_{m,n}\rangle \langle\psi_{m,n}| \quad (1.139)$$

$$\rho_B = \text{Tr}_{AC}(|\chi\rangle_{ABC}\langle\chi|) = \sum_{m,n=0}^{d-1} |\beta_{m,n}|^2 |\psi_{m,n}\rangle \langle\psi_{m,n}| \quad (1.140)$$

where

$$|\psi_{m,n}\rangle = U_{m,n}|\psi\rangle \quad (1.141)$$

Furthermore, Ghiu [75] studied and analyzed the class of universal asymmetric cloning machines for d-level systems. These cloning machines are universal in the sense that they generate the outputs which are independent of the input state. He also showed that the universal cloning machine is optimal in the sense that a cloning machine creates one clone with maximal fidelity for a given fidelity of the other clone.

The optimal universal asymmetric Heisenberg cloning machine is given by

$$U|j\rangle|00\rangle = \frac{1}{\sqrt{1+(d-1)(p^2+q^2)}}(|j\rangle|j\rangle|j\rangle + p \sum_{s=1}^{d-1} |j\rangle|(j+s) \text{ modulo } d\rangle \otimes \\ |(j+s) \text{ modulo } d\rangle + q \sum_{s=1}^{d-1} |(j+s) \text{ modulo } d\rangle|j\rangle|(j+s) \text{ modulo } d\rangle) \quad (1.142)$$

where $|j\rangle$ is the computational basis, $\alpha_{m,n} = \mu$, $\forall (m,n) \neq (0,0)$, $\alpha_{0,0} = \nu$, $p = \frac{(\nu-\mu)}{[\nu+(d-1)\mu]}$ and $q = \frac{d\mu}{[\nu+(d-1)\mu]} = 1 - p$.

After operating optimal universal asymmetric Heisenberg cloning transformation on the input state $|\psi\rangle = \sum_{j=0}^{d-1} \alpha_j |j\rangle$, the output states are described by the density operators

$$\rho_A = \frac{1}{\sqrt{1+(d-1)(p^2+q^2)}} \{ [1 - q^2 + (d-1)p^2] |\psi\rangle\langle\psi| + q^2 I \} \quad (1.143)$$

$$\rho_B = \frac{1}{\sqrt{1+(d-1)(p^2+q^2)}} \{ [1 - p^2 + (d-1)q^2] |\psi\rangle\langle\psi| + p^2 I \} \quad (1.144)$$

To quantify the quality of the copies produced from universal asymmetric Heisenberg cloning machine, the fidelities are to be calculated. The fidelities of the two non-identical clones described by the density operators ρ_A and ρ_B are given by

$$F_A = \langle\psi|\rho_A|\psi\rangle = \frac{1 + (d-1)p^2}{1 + (d-1)(p^2+q^2)} \quad (1.145)$$

$$F_B = \langle\psi|\rho_B|\psi\rangle = \frac{1 + (d-1)q^2}{1 + (d-1)(p^2+q^2)} \quad (1.146)$$

The universal asymmetric Heisenberg cloning machine produces the best quality copies when $F_A + F_B$ takes the maximum value. It can be shown that the maximum value of

$F_A + F_B$ will be attained when $p = q = \frac{1}{2}$. Inserting $p = q = \frac{1}{2}$ in equations (1.145-1.146), we get

$$F_A = F_B = \frac{d+3}{2(d+1)} \quad (1.147)$$

Therefore, equation (1.142) represents the general expression for the optimal universal asymmetric Heisenberg cloning machine and it reduces to optimal universal symmetric cloning machine when $p = q = \frac{1}{2}$.

1.5 Quantum cloning of mixed state

In this section we want to focus on the approximate copying of mixed state. In quantum cryptography, the sender encodes the information into two non-orthogonal pure states. Then the information is communicated through a communication channel. Actually, in reality a communication channel will inevitably suffer from noise that will have caused the pure states to evolve to mixed states. If the third party (Eve) intercepts the message in a midway and wants to extract the information encoded in a message without revealing her presence to the sender and receiver, she has to clone the intercepted mixed state with maximum possible accuracy. Hence the approximate cloning of mixed state is interesting and important in the field of quantum cryptography.

In 2003, A.E.Rastegin [130] defined the global fidelity for mixed states in the same way as for pure states. The global fidelity for mixed state cloning for the set $\{\rho_1, \rho_2\}$ can be defined as

$$F_G^M = \frac{1}{2}[F(\tilde{\rho}_1, \rho_1 \otimes \rho_1) + F(\tilde{\rho}_2, \rho_2 \otimes \rho_2)] \quad (1.148)$$

where $\tilde{\rho}_1, \tilde{\rho}_2$ denotes the output of the system respectively.

He also found the upper bound of the global fidelity F_G^M of mixed state cloning for the set $S = \{\rho_1, \rho_2\}$:

$$\text{Upper bound of } F_G^M = \frac{1}{2}[1 + f^3 + (1 - f^2)\sqrt{1 + f^2}] \quad (1.149)$$

where $f = \sqrt{F(\rho_1, \rho_2)}$.

If both states to be cloned are pure, then the definition of global fidelity of mixed state

reduces to the definition of global fidelity of pure state and also the upper bound of the fidelity F_G^M coincides with the upper bound of the fidelity for pure state cloning. But the state dependent cloner for pure state cloning cannot be used for mixed state cloning because of the two reasons: (i) State dependent cloning transformation requires no auxiliary system and (ii) The initial state of the copy mode is pure. Further, Rastegin [131] extended the known global-fidelity limits of state-dependent cloning to mixed quantum states. He [129] also extend the concept of the relative error to the mixed-state cloning and obtained a lower bound of it. He had also shown that the lower bound of the relative error contributes to the stronger no-cloning theorem [97].

H.Fan [62] proposed a quantum cloning machine for arbitrary mixed states in symmetric subspace and showed that the introduced quantum cloning machine can be used to copy part of the output state of another quantum cloning machine (e.g. Gisin-Massar $1 \rightarrow 3$ quantum cloning machine) and is useful in quantum computation and quantum information. His proposed 2 to 3 quantum quantum cloning machine for mixed state is given by

$$U|2 \uparrow\rangle \otimes R = \frac{\sqrt{3}}{2}|3 \uparrow\rangle \otimes R_{\uparrow} + \frac{1}{2}|2 \uparrow, \downarrow\rangle \otimes R_{\downarrow} \quad (1.150)$$

$$U|2 \downarrow\rangle \otimes R = \frac{1}{2}|\uparrow, 2 \downarrow\rangle \otimes R_{\uparrow} + \frac{\sqrt{3}}{2}|\downarrow\rangle \otimes R_{\downarrow} \quad (1.151)$$

$$U|\uparrow, \downarrow\rangle \otimes R = \frac{1}{\sqrt{2}}|2 \uparrow, \downarrow\rangle \otimes R_{\uparrow} + \frac{1}{\sqrt{2}}|\uparrow, 2 \downarrow\rangle \otimes R_{\downarrow} \quad (1.152)$$

where state $|2 \uparrow, \downarrow\rangle$ is a normalized symmetric state with 2 spin up and 1 spin down.

The quantum cloning machine (1.150-1.152) produced three copies of the 2-qubit mixed state with fidelity $\frac{79}{108}$.

Recently, Fan, Liu and Shi [63] studied the quantum cloning of two identical mixed qubits $\rho \otimes \rho$. They proposed a general quantum cloning machine which creates M copies from 2 identical mixed qubits. Their quantum cloning transformation is given by

$$U|\uparrow\uparrow\rangle \otimes R = \sum_{k=0}^{M-2} \alpha_{0k} |(M-k) \uparrow, k \downarrow\rangle \otimes R_k \quad (1.153)$$

$$U|\downarrow\downarrow\rangle \otimes R = \sum_{k=0}^{M-2} \alpha_{2k} |(M-2-k)\uparrow, (2+k)\downarrow\rangle \otimes R_k \quad (1.154)$$

$$\frac{1}{\sqrt{2}}U(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \otimes R = \sum_{k=0}^{M-2} \alpha_{1k} |(M-1-k)\uparrow, (1+k)\downarrow\rangle \otimes R_k \quad (1.155)$$

$$\frac{1}{\sqrt{2}}U(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \otimes R = \sum_{k=0}^{M-2} \alpha_{1k} |(\widetilde{M-1-k})\uparrow, (1+k)\downarrow\rangle \otimes R_k \quad (1.156)$$

where

$$\alpha_{jk} = \sqrt{\frac{6(M-2)!(M-j-k)!(j+k)!}{(2-j)!(M+1)!(M-2-k)!j!k!}}, \quad j = 0, 1, 2 \quad (1.157)$$

The state $|i\uparrow, j\downarrow\rangle$ is a completely symmetrical state with i spins up and j spins down, the state $|\widetilde{i\uparrow}, j\downarrow\rangle$ is orthogonal to $|i\uparrow, j\downarrow\rangle$. R_k represents the final orthogonal machine states.

Each output of the quantum cloning machine is described by the reduced density operator

$$\rho_{red.}^{(out)} = \frac{M+2}{2M}\rho + \frac{M-2}{4M}I \quad (1.158)$$

Since the reduction factor $\frac{M+2}{2M}$ achieves the optimal bound so the quantum cloning machine (1.153-1.157) copies the two identical mixed qubits and two identical pure states optimally.

1.6 Quantum cloning and no-signalling

The non-local property of the quantum mechanics cannot be used for superluminal signalling. This fact has already been vividly discussed in the past [73, 74, 88, 110, 124, 134, 147]. If a perfect quantum cloning machine were available, Bob could generate an infinite number of copies of his state, and therefore would be able to determine his state with perfect accuracy, thus knowing what basis Alice decided to use. In this way, transfer of information between Alice and Bob would be possible. In particular, if they are space-like separated, information could be transmitted with superluminal speed,

but no-cloning theorem ruled out this possibility. However, imperfect quantum cloning machines exist [20, 28, 79, 112]. So, naturally a question arises, whether approximate quantum cloning machine can make possible the superluminal signalling or not? In 1998, Gisin [81] first attacked this problem and showed that any approximate optimal quantum cloning cannot lead to signalling. Therefore the construction of optimal quantum cloning machine does not violate the "peaceful coexistence" between quantum mechanics and relativity. He used the no-signalling constraint to derive a bound on the fidelity of quantum cloning machine and showed that this bound coincides with the fidelity of the Buzek-Hillery universal quantum cloning machine. This result again proves that Buzek-Hillery universal quantum cloning machine is optimal. After this work, many work had been done on quantum cloning and signalling. Ghosh, Kar and Roy [76] used the no-signalling constraint to find the optimality of the universal asymmetric quantum cloning machine of Buzek, Hillery and Bednik [32].

Bruss, Ariano, Macchiavello and Sacchi [22] showed that any linear trace-preserving map forbids superluminal signalling but converse is not true. The no-signalling condition implies only linearity but does not imply the two important properties of quantum operations namely positivity and trace-preservation. Hence, there exist some maps that go beyond quantum mechanics, but still preserve the constraint of no-superluminal signalling. They also gave an example to explain the fact that the cloning fidelity is unrelated to the no-signalling condition and hence any bound on a cloning fidelity cannot be derived from the no-signalling constraint alone. Quantum mechanics as a complete theory, guarantees no- superluminal signalling, and gives the correct known upper bounds on the fidelity of quantum cloning.

Duan and Guo [50] introduced a probabilistic quantum cloning machine which can be used to produce the perfect clones of the quantum states secretly chosen from a certain set of linearly independent states, with some probability less than unity. Although probabilistic quantum cloning machine produces perfect clones but it cannot be used for superluminal signalling [117]. Hardy and Song [86] showed that no-signalling condition

lead to the constraints on probabilistic quantum cloning machine i.e. if probabilistic quantum cloning machine produces exact clones of $(d+1)$ number of quantum states, in which d number of states are linearly independent, then there will be signalling. Further, Ghosh, Kar, Kunkri and Roy [77] used the technique of remote state preparation to prove the Hardy and Song's result in a more simpler way. They showed that probabilistic exact cloning of any three different states of a qubit implies (probabilistic) signalling in the sense, that one can extract more than 1 classical bit message probabilistically by communicating 1 classical bit only. They also generalized this result in d -dimensional Hilbert space.

1.7 *Quantum Deletion machine*

A.K.Pati and S.L.Braunstein [118] were the first to observe the fact that it is not possible to delete the information content of one or more photons by a physical process. That is, the linearity of quantum theory forbids deleting one unknown quantum state against a copy in either a reversible or an irreversible manner. This phenomenon is called "quantum no-deleting" principle. This principle is complementary to the "quantum no-cloning theorem". If quantum deleting could be done, then one would create a standard blank state onto which one could copy an unknown state approximately, by deterministic cloning or exactly, by probabilistic cloning. Therefore, when memory in a quantum computer is scarce, quantum deleting may play an important role, and one could store new information in an already computed state by deleting the old information.

We can understand the principle behind quantum deletion more clearly, if we compare quantum deletion with the "Landauer erasure principle" [103]. It tells us that a single copy of some classical information can be erased at some energy cost. It is an irreversible operation. In quantum information theory, erasure of a single unknown state may be thought of as swapping it with some standard state and then dumping it into the environment. Unlike quantum erasure, quantum deletion is a different concept. Quantum

deletion is more like reversible 'uncopying' of an unknown quantum state. The essential difference is that irreversible erasure naturally carries over from the classical to the quantum world, whereas the analogous uncopied of classical information is impossible for quantum information. Pati and Braunstein [121] had shown that the violation of no-deletion principle can lead to superluminal signalling using non-local entangled states. Therefore, no-deletion principle supports the "peaceful co-existence" between quantum mechanics and relativity. However, the (Landauer) erasure of information does not allow for any signalling. This fact provides another evidence in support of the statement that quantum deletion is fundamentally a different operation than erasure.

Although there is not a perfect deleting machine, the corresponding no-deleting principle does not prohibit us from constructing the approximate deleting machine. Pati et.al. [119] studied the distribution of the quantum information among various subsystems during the deletion process. They introduced a state dependent, approximate quantum deletion machine and named it as conditional deletion machine.

The conditional deletion machine (deletion transformation) for orthogonal qubits is defined by

$$|0\rangle|0\rangle|A\rangle \rightarrow |0\rangle|\Sigma\rangle|A_0\rangle \quad (1.159)$$

$$|1\rangle|1\rangle|A\rangle \rightarrow |1\rangle|\Sigma\rangle|A_1\rangle \quad (1.160)$$

$$|0\rangle|1\rangle|A\rangle \rightarrow |0\rangle|1\rangle|A\rangle \quad (1.161)$$

$$|1\rangle|0\rangle|A\rangle \rightarrow |1\rangle|0\rangle|A\rangle \quad (1.162)$$

where $|A\rangle$ is the initial state and $|A_0\rangle, |A_1\rangle$ are the final states of ancilla and they are mutually orthogonal to each other. The speciality of the introduced deletion machine lies in the fact that if the two input qubits are identical then it deletes a copy but if they are different then it allows them to pass through without any change.

Let

$$|\Psi\rangle = \alpha|0\rangle + \beta|1\rangle \quad (1.163)$$

with $|\alpha|^2 + |\beta|^2 = 1$ be any arbitrary quantum state.

Each of a copy from two copies of an arbitrary quantum state $|\psi\rangle$ can be approximately deleted by the deletion transformation (1.159-1.162) and it will create the following state

$$\begin{aligned} |\Psi\rangle_a |\Psi\rangle_b |A\rangle_c &\rightarrow \alpha^2 |0\rangle_a |\Sigma\rangle_b |A_0\rangle_c + \beta^2 |1\rangle_a |\Sigma\rangle_b |A_1\rangle_c + \alpha\beta(|01\rangle_{ab} + |10\rangle_{ab})|A\rangle_c \\ &= |\Psi_{out}\rangle_{abc}. \end{aligned} \quad (1.164)$$

The reduced density matrix of the two qubits ab after the deletion operation is given by

$$\begin{aligned} \rho_{ab} = \text{tr}_c(|\Psi_{out}\rangle_{abc}\langle\Psi_{out}|) &= |\alpha|^4 |0\rangle\langle 0| \otimes |\Sigma\rangle\langle\Sigma| + |\beta|^4 |1\rangle\langle 1| \otimes |\Sigma\rangle\langle\Sigma| + \\ &2|\alpha|^2 |\beta|^2 |\psi^+\rangle\langle\psi^+| \end{aligned} \quad (1.165)$$

where $|\psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$.

The reduced density matrix for the qubit in the mode 'a' and 'b' respectively are

$$\rho_a = \text{tr}_b(\rho_{ab}) = |\alpha|^4 |0\rangle\langle 0| + |\beta|^4 |1\rangle\langle 1| + |\alpha|^2 |\beta|^2 I \quad (1.166)$$

$$\rho_b = \text{tr}_a(\rho_{ab}) = (1 - 2|\alpha|^2 |\beta|^2) |\Sigma\rangle\langle\Sigma| + |\alpha|^2 |\beta|^2 I \quad (1.167)$$

The fidelity of the qubit in mode 'a' is given by

$$F_a = \langle\Psi|\rho_a|\Psi\rangle = 1 - 2|\alpha|^2 |\beta|^2 \quad (1.168)$$

When $\alpha = \beta = \frac{1}{\sqrt{2}}$, i.e. for an equal superposition of qubit state, the fidelity of the qubit in mode 'a' reduces to $\frac{1}{2}$. The average fidelity is $\overline{F}_a = \frac{2}{3} \approx 0.66$. This shows that the first mode of the qubit is not faithfully retained during the deletion operation.

The fidelity of deletion is given by

$$F_b = \langle\Sigma|\rho_b|\Sigma\rangle = 1 - |\alpha|^2 |\beta|^2 \quad (1.169)$$

For an equal superposition of qubit state the fidelity of deletion takes the value $\frac{3}{4}$ which is the maximum limit for deleting an unknown qubit. Furthermore, we can observe that for a classical bit, i.e. for either $\alpha = 0$ and $\beta = 1$ or $\alpha = 1$ and $\beta = 0$, the qubit in mode 'b' is perfectly deleted and simultaneously the deletion machine faithfully retained the qubit in mode 'a'.

Since the fidelity of deletion F_b depends on the input state so it is important to calculate the average fidelity and it is given by $\overline{F_b} = \frac{5}{6} \approx 0.83$.

As we see that quantum deletion machine introduced by Pati et.al. is state dependent, so it is natural to ask the question as in the case of quantum cloning machines, whether there exists any quantum deletion machine which works in a similar fashion for all arbitrary input states? D.Qiu [126] was the first who attempted to answer the above question and got success partially. He verified that some standard universal quantum deleting machine does not exist. Not only that he constructed a universal deletion machine but unfortunately the machine was found to be non optimal in the sense of low fidelity of deletion.

A non-optimal universal quantum deletion machine [126] is defined by

$$U|0\rangle|0\rangle|Q\rangle = \frac{1}{\sqrt{2}}|0\rangle|A\rangle + \frac{1}{\sqrt{2}}|1\rangle|B\rangle \quad (1.170)$$

$$U|1\rangle|1\rangle|Q\rangle = \frac{1}{\sqrt{2}}i|1\rangle|B\rangle - \frac{1}{\sqrt{2}}i|0\rangle|A\rangle \quad (1.171)$$

$$U|0\rangle|1\rangle|Q\rangle = |0\rangle|1\rangle \quad (1.172)$$

$$U|1\rangle|0\rangle|Q\rangle = |1\rangle|0\rangle \quad (1.173)$$

where for any real numbers r_1, r_2 with $r_1^2 + r_2^2 = 1$, if $|A\rangle = r_1|0\rangle + r_2|1\rangle$, then $|B\rangle = r_2|0\rangle - r_1|1\rangle$.

Using deletion machine (1.170-1.173), one can delete a copy of a qubit from two identical copies with fidelity $\frac{1}{2}$. Although the fidelity of deletion is input state independent but its value does not give any satisfactory result. Therefore, the prescribed deletion machine (1.170-1.173) is universal but it is not an optimal one. Recently, we designed a universal quantum deletion machine which improves the fidelity of deletion from 0.5 and takes it to 0.75 in the limiting sense [2].

Also W.Song, M.Yang and Z-L Cao [137] constructed a state dependent quantum deleting machine without considering the ancilla.

It is described by the following unitary transformation

$$U|\psi_i^N\rangle = |\phi_i\rangle|\Sigma\rangle^{\otimes(N-M)} \quad (1.174)$$

where $|\psi_i^N\rangle$ are the N-fold tensor product states $|\psi_i^N\rangle = |\psi_i\rangle_1 \otimes \dots \otimes |\psi_i\rangle_N$ which are prepared in the same state, and $|\psi_i\rangle$ is chosen from a set of K non-orthogonal states, $|\phi_i\rangle$ is the output state after the machine deleting $|\psi_i\rangle^{\otimes(N-M)}$. The global fidelity which characterizes the distance between the output state $|\phi_i\rangle$ and the ideal state $|\psi_i\rangle^{\otimes(M)}$ is defined by

$$F = \sum_{i=1}^K \eta_i |\langle \psi_i^M | \phi_i \rangle|^2 \quad (1.175)$$

where η_i denotes a priori probability of the state $|\psi_i\rangle^{\otimes(N)}$.

They found the optimal value of the global fidelity when K=2 and it is given by

$$F^{(opt)} = \frac{1}{2} \{1 + [1 - 4\eta_1\eta_2 \sin^2(2\theta - \varphi_1 + \varphi_2)]^{\frac{1}{2}}\}. \quad (1.176)$$

where $\eta_1 + \eta_2 = 1$ and $|\psi_1^M\rangle = \cos\theta|\alpha\rangle + \sin\theta|\beta\rangle$, $|\psi_2^M\rangle = \cos\theta|\alpha\rangle - \sin\theta|\beta\rangle$, $|\phi_1^M\rangle = \cos\varphi_1|\alpha\rangle + \sin\varphi_1|\beta\rangle$, $|\phi_2^M\rangle = \cos\varphi_2|\alpha\rangle + \sin\varphi_2|\beta\rangle$. The states $|\alpha\rangle$ and $|\beta\rangle$ are orthonormal basis for the subspace spanned by $|\psi_1^M\rangle$ and $|\psi_2^M\rangle$.

The optimal global fidelity can attain the value one when one of the a priori probabilities is zero.

The possibility of perfect deletion with some probability less than one cannot be ruled out [65, 127]. J. Feng, Y-F Gao, J-S Wang and M-S Zhan [65] designed a probabilistic quantum deletion machine and showed that each of the two copies of non-orthogonal and linearly independent quantum states can be probabilistically deleted by a general unitary-reduction operation. Their prescribed quantum deletion machine can be described by the following unitary operation U [65]:

$$U(|\psi_i\rangle|\psi_i\rangle|P_0\rangle) = \sqrt{b_i}|\psi_i\rangle|\Sigma\rangle|P_i\rangle + \sum_{l=1}^{k^2} \sqrt{f_i^{(l)}}|\mu_l\rangle|P_0\rangle \quad (i = 1, 2, \dots, k) \quad (1.177)$$

where $|\psi_i\rangle|\psi_i\rangle$ (i=1,2,...,k) are the input normalized states of the system D which belongs to a Hilbert space of dimension k^2 , and $|\mu_l\rangle$ ($l = 1, 2, \dots, k^2$) are the orthonormal basis states of above space. $|\Sigma\rangle$ is the normalized standard blank state in Hilbert space of dimension k and $|P_i\rangle$ (i=0,1,2,...,k) are normalized states of the probe system P with a k_p -dimensional Hilbert space ($k_p \geq k + 1$). $|P_0\rangle, |P_1\rangle, |P_2\rangle, \dots, |P_k\rangle$ are not generally

orthogonal, but each of $|P_i\rangle$ ($i=1,2,\dots,k$) is orthogonal to $|P_0\rangle$.

They also generalized the results of $2 \rightarrow 1$ probabilistic deleting to the case of $N \rightarrow M$ deleting (N, M are positive integers and $N > M$).

Chapter 2

Hybrid quantum cloning

Insofar as mathematics is about reality, it is not certain, and insofar as it is certain, it is not about reality - Albert Einstein

Quantum mechanics and relativity, taken together, are extraordinarily restrictive, and they therefore provide us with a great logical machine. We can explore with our minds any number of possible universes consisting of all kinds of mythical particles and interactions, but all except a very few can be rejected on a priori grounds because they are not simultaneously consistent with special relativity and quantum mechanics. Hopefully in the end we will find that only one theory is consistent with both and that theory will determine the nature of our particular universe - Steven Weinberg

2.1 *Prelude*

A fundamental restriction in quantum theory is that quantum information cannot be copied perfectly [147] in contrast to the information we talk about in classical world. Similarly, it is known that quantum information cannot be deleted against a copy. But if we pay some price, then approximate or probabilistic cloning [25, 39, 44, 50, 113, 132, 133, 155] and deletion operations [1, 2, 4, 118, 128] are possible. For example, it does not prohibit the possibility of approximate cloning of an arbitrary state of a quantum mechanical system. The existence of 'Universal Copying Machine' (UCM) created a class of approximate cloning machines which are independent of the input state

[23, 28, 30, 79, 150]. The optimality of such cloning transformations has been verified [79]. There also exists another class of copying machines which are state dependent. The original proof of the no-cloning theorem was based on the linearity of the evolution. Later it was shown that the unitarity of quantum theory also forbids us from accurate cloning of non-orthogonal states with certainty [45, 151]. But as discussed in section-1.4.3, non-orthogonal states secretly chosen from a set can be faithfully cloned with certain probabilities [50, 51] or can evolve into a linear superposition of multiple-copy states together with a failure term described by a composite state [116] if and only if the states are linearly independent. In quantum world it is very important to know various limitations imposed by quantum theory on quantum information. Recently, some general impossible operations have been studied by Pati [120] in detail. This unifies the no-cloning, no-complementing and no-conjugating theorems in quantum information theory. Among all the impossible operations [33, 118, 147, 157], the impossibility of 'cloning-cum-complementing' quantum machines attracts much attention here in the sense that it is a combination of cloning machine and complementing machine where the probabilities of separately existing cloning machines are λ and $1 - \lambda$, respectively. In the same spirit, we can imagine a hybrid cloning machine which is a superposition of two cloning machines [120]. One can construct hybrid cloning machine by combining different existing cloning transformations. Hybrid quantum cloning machines can be divided into two groups: (i) State dependent and (ii) State independent or Universal. The main objective of this chapter is to study the behavior of such types of hybrid cloning machines.

Before going into the discussion about the hybrid quantum cloning machine, we would like to discuss briefly about universal asymmetric Pauli cloning machine and universal anti-cloning machine.

Universal asymmetric Pauli cloning machine: Asymmetric cloning transformation

[36, 37] is given by

$$|0\rangle|\Sigma\rangle|Q\rangle \longrightarrow \left(\frac{1}{\sqrt{1+p^2+q^2}}\right)(|0\rangle|0\rangle|\uparrow\rangle + (p|0\rangle|1\rangle + q|1\rangle|0\rangle)|\downarrow\rangle, \quad (2.1)$$

$$|1\rangle|\Sigma\rangle|Q\rangle \longrightarrow \left(\frac{1}{\sqrt{1+p^2+q^2}}\right)(|1\rangle|1\rangle|\downarrow\rangle + (p|1\rangle|0\rangle + q|0\rangle|1\rangle)|\uparrow\rangle. \quad (2.2)$$

where $p + q = 1$.

Pauli cloning machines (transformations) are nothing but asymmetric cloning machines that generate two non-identical approximate copies of a single quantum bit, each output qubits emerging from a Pauli channel (discussed in subsection 1.4.6) [36]. The asymmetric quantum cloning machine play an important role in the situation in which one of the clones need to be a bit better than the other.

Table-2.1: Fidelity of the copies produced from asymmetric Pauli cloning machine

parameter (p)	$(F_1)_{PCM} = \frac{(p^2+1)}{2(p^2-p+1)}$	$(F_2)_{PCM} = \frac{(p^2-2p+2)}{2(p^2-p+1)}$	Difference between qualities of the two copies $(F_1)_{PCM} \sim (F_2)_{PCM}$
0.0	0.50	1.00	0.50
0.1	0.55	0.99	0.44
0.2	0.62	0.98	0.36
0.3	0.69	0.94	0.25
0.4	0.76	0.89	0.13
0.5	0.83	0.83	0.00 (Symmetric copies)
0.6	0.89	0.76	0.13
0.7	0.94	0.69	0.25
0.8	0.98	0.62	0.36
0.9	0.99	0.55	0.44
1.0	1.00	0.50	0.50

Illustration of the Table 2.1:

The above table represents the quality of the two different outputs from asymmetric Pauli cloning machine in terms of the fidelity for different values of the parameter p . We find that when $p = 0$ or $p = 1$, one of the output is totally undisturbed i.e. it contains the whole information of the input quantum state while the overlapping of the other output with the original is found to be 0.5. For $p = 0.5$, the Pauli cloning machine reduces to B-H symmetric quantum cloning machine. We also observe here that

the Pauli quantum cloning machine gives better quality asymmetric copies when $p = 0.4$ and $p = 0.6$ because the difference between the quality of the copies is small in these cases.

Universal anti-cloning machine: Few years earlier, Gisin and Popescu [82] discovered an important fact that quantum information is better stored in two anti-parallel spins as compared to two parallel spins. This fact gave birth to a new type of cloning machine called anti-cloning machine [82, 136] which generates two outputs, one of the output has the same direction as the input and the other output has direction opposite to the input. Song and Hardy [136] constructed a universal quantum anti-cloner which takes an unknown quantum state just as in quantum cloner but its output as one with the same copy while the second one with opposite spin direction to the input state. For the Bloch vector, an input \mathbf{n} , quantum anti-cloner would have the input as $\frac{1}{2}(\mathbf{1} + \mathbf{n} \cdot \boldsymbol{\sigma})$, then it generates two outputs, $\frac{1}{2}(\mathbf{1} + \eta \mathbf{n} \cdot \boldsymbol{\sigma})$ and $\frac{1}{2}(\mathbf{1} - \eta \mathbf{n} \cdot \boldsymbol{\sigma})$, where $0 \leq \eta \leq 1$ is the shrinking factor and the fidelity is defined as $F = \langle \mathbf{n} | \rho^{out} | \mathbf{n} \rangle = \frac{1}{2}(1 + \eta)$. If spin flipping were allowed then anti-cloner would have the same fidelity as the regular cloner since one could clone first then flip the spin of the second copy. However spin flipping of an unknown state is not allowed in quantum mechanics. They also showed that the quantum state can be anti-cloned exactly with non-zero probability.

The universal anti-cloning transformation is given by

$$|0\rangle|\Sigma\rangle|Q\rangle \longrightarrow \frac{1}{\sqrt{6}}|0\rangle|0\rangle|\uparrow\rangle + \left(\left(\frac{1}{\sqrt{2}}\right)e^{icos^{-1}(\frac{1}{\sqrt{3}})}|0\rangle|1\rangle - \frac{1}{\sqrt{6}}|1\rangle|0\rangle\right)|\rightarrow\rangle + \frac{1}{\sqrt{6}}|1\rangle|1\rangle|\leftarrow\rangle, \quad (2.3)$$

$$|1\rangle|\Sigma\rangle|Q\rangle \longrightarrow \frac{1}{\sqrt{6}}|1\rangle|1\rangle|\rightarrow\rangle + \left(\left(\frac{1}{\sqrt{2}}\right)e^{icos^{-1}(\frac{1}{\sqrt{3}})}|1\rangle|0\rangle - \frac{1}{\sqrt{6}}|0\rangle|1\rangle\right)|\uparrow\rangle + \frac{1}{\sqrt{6}}|0\rangle|0\rangle|\downarrow\rangle, \quad (2.4)$$

where $|\uparrow\rangle, |\downarrow\rangle, |\rightarrow\rangle, |\leftarrow\rangle$ are orthogonal machine states. The fidelity of universal anti-cloner is same as the fidelity of measurement which is equal to $\frac{2}{3}$ [108].

In the subsequent sections, we will discuss about the state dependent and state inde-

pendent hybrid quantum cloning transformations. This chapter is based on our work "Hybrid quantum cloning machine".

2.2 State dependent hybrid cloning transformation

In this section, we study two state dependent hybrid quantum cloning machines. The quality of the state dependent cloning machine depends on the input state so naturally one may ask a question why this type of cloning machine is important for study? Here we give a reason for such studies. The importance of the state dependent cloner lies in the eavesdropping strategy on some quantum cryptographic system. For example, if the quantum key distribution protocol is based on two non-orthogonal states [16], the optimal state dependent cloner can clone the qubit in transit between a sender and a receiver. The original qubit can then be re-sent to the receiver and the clone can stay with an eavesdropper who by measuring it can obtain some information about the bit value encoded in the original. The eavesdropper may consider storing the clone and delaying the actual measurement until any further public communication between the sender and the receiver takes place. This eavesdropping strategy has been discussed in [20, 80].

B-H type cloning transformation: B-H cloning transformation generally indicates the optimal universal quantum cloning transformation but in this paper, we relax one condition of universality of B-H cloning transformation and hence we rename the B-H cloning transformation as B-H type cloning transformation. Therefore, although B-H type cloning transformation is structurally same as the universal B-H cloning transformation but it is different in the sense that this type of transformation is state dependent. State dependent ness of the cloning machine arises because of the relaxation of the condition $\frac{\partial D_{ab}}{\partial \alpha^2} = 0$.

2.2.1 Hybridization of two B-H type cloning transformation:

Here we investigate a new kind of cloning transformation that can be obtained by combining two different BH type cloning transformations. Here we consider two B-H type cloning transformations which occur separately in the hybrid cloning transformation with probability λ and $1 - \lambda$ respectively.

The hybrid quantum cloning transformation can be written as

$$\begin{aligned} |\psi\rangle|\Sigma\rangle|Q\rangle \otimes |n\rangle \longrightarrow & \sqrt{\lambda}[|\psi\rangle|\psi\rangle|Q_\psi\rangle + (|\psi\rangle|\bar{\psi}\rangle + |\bar{\psi}\rangle|\psi\rangle)|Y_\psi\rangle]|i\rangle \\ & + (\sqrt{1-\lambda})[|\psi\rangle|\psi\rangle|Q'_\psi\rangle + (|\psi\rangle|\bar{\psi}\rangle + |\bar{\psi}\rangle|\psi\rangle)|Y'_\psi\rangle]|j\rangle. \end{aligned} \quad (2.5)$$

Unitarity of the transformation gives

$$\lambda(\langle Q_\psi|Q_\psi\rangle + 2\langle Y_\psi|Y_\psi\rangle) + (1-\lambda)(\langle Q'_\psi|Q'_\psi\rangle + 2\langle Y'_\psi|Y'_\psi\rangle) = 1, \quad (2.6)$$

$$2\lambda(\langle Y_\psi|Y_{\bar{\psi}}\rangle) + 2(1-\lambda)(\langle Y'_\psi|Y'_{\bar{\psi}}\rangle) = 0. \quad (2.7)$$

Equations (2.6) and (2.7) are satisfied for all values of λ ($0 < \lambda < 1$) if

$$\langle Q_\psi|Q_\psi\rangle + 2\langle Y_\psi|Y_\psi\rangle = \langle Q'_\psi|Q'_\psi\rangle + 2\langle Y'_\psi|Y'_\psi\rangle = 1 \quad (2.8)$$

$$\langle Y_\psi|Y_{\bar{\psi}}\rangle = \langle Y'_\psi|Y'_{\bar{\psi}}\rangle = 0 \quad (2.9)$$

Further we assume that

$$\langle Q_\psi|Y_\psi\rangle = 0 = \langle Q_\psi|Q_{\bar{\psi}}\rangle. \quad (2.10)$$

Let $|\chi\rangle = \alpha|0\rangle + \beta|1\rangle$ with $\alpha^2 + \beta^2 = 1$, be the input state. The cloning transformation (2.5) approximately copies the information contained in the input state $|\chi\rangle$ into two identical states described by the density operators $\rho_a^{(out)}$ and $\rho_b^{(out)}$, respectively. The reduced density operator $\rho_a^{(out)}$ is given by

$$\begin{aligned} \rho_a^{(out)} = & |0\rangle\langle 0|[\alpha^2 + (\beta^2\langle Y'_1|Y'_1\rangle - \alpha^2\langle Y'_0|Y'_0\rangle) + \lambda(\beta^2\langle Y_1|Y_1\rangle - \alpha^2\langle Y_0|Y_0\rangle - \beta^2\langle Y'_1|Y'_1\rangle + \\ & \alpha^2\langle Y'_0|Y'_0\rangle)] + |0\rangle\langle 1|[\alpha\beta(\langle Q'_1|Y'_0\rangle + \langle Y'_1|Q'_0\rangle) + \lambda\alpha\beta(\langle Q_1|Y_0\rangle + \langle Y_1|Q_0\rangle - \\ & \langle Q'_1|Y'_0\rangle - \langle Y'_1|Q'_0\rangle)] + |1\rangle\langle 0|[\alpha\beta(\langle Q'_1|Y'_0\rangle + \langle Y'_1|Q'_0\rangle) + \lambda\alpha\beta(\langle Q_1|Y_0\rangle + \\ & \langle Y_1|Q_0\rangle - \langle Q'_1|Y'_0\rangle - \langle Y'_1|Q'_0\rangle)] + |1\rangle\langle 1|[\beta^2 - (\beta^2\langle Y'_1|Y'_1\rangle - \alpha^2\langle Y'_0|Y'_0\rangle) + \\ & \lambda(\beta^2\langle Y_1|Y_1\rangle - \alpha^2\langle Y_0|Y_0\rangle - \beta^2\langle Y'_1|Y'_1\rangle + \alpha^2\langle Y'_0|Y'_0\rangle)]. \end{aligned} \quad (2.11)$$

The other output state described by the density operator $\rho_b^{(out)}$ looks exactly the same as $\rho_a^{(out)}$.

Let $\langle Y_0|Y_0\rangle = \langle Y_1|Y_1\rangle = \xi$, $\langle Q_1|Y_0\rangle = \langle Y_0|Q_1\rangle = \langle Q_0|Y_1\rangle = \langle Y_1|Q_0\rangle = \frac{\eta}{2}$,
 $\langle Y'_0|Y'_0\rangle = \langle Y'_1|Y'_1\rangle = \xi'$ and $\langle Q'_1|Y'_0\rangle = \langle Y'_0|Q'_1\rangle = \langle Q'_0|Y'_1\rangle = \langle Y'_1|Q'_0\rangle = \frac{\eta'}{2}$
 with $0 \leq \xi(\xi') \leq 1$ and $0 \leq \eta(\eta') \leq 2\sqrt{\xi(1-2\xi)}(2\sqrt{\xi'(1-2\xi')}) \leq \frac{1}{\sqrt{2}}$.

Using above conditions, equation (2.11) can be rewritten as

$$\begin{aligned} \rho_a^{(out)} = & |0\rangle\langle 0|[\alpha^2 + \xi'(\beta^2 - \alpha^2) + \lambda(\xi - \xi')(\beta^2 - \alpha^2)] + |0\rangle\langle 1|[\alpha\beta(\eta' + \lambda(\eta - \eta'))] \\ & + |1\rangle\langle 0|[\alpha\beta(\eta' + \lambda(\eta - \eta'))] + |1\rangle\langle 1|[\beta^2 - \xi'(\beta^2 - \alpha^2) - \lambda(\xi - \xi')(\beta^2 - \alpha^2)]. \end{aligned} \quad (2.12)$$

To investigate how well our hybrid cloning machine copies the input state, we have to calculate the fidelity. The fidelity F_{HCM} is defined by

$$\begin{aligned} F_{HCM} = \langle \chi | \rho_a^{(out)} | \chi \rangle = & \alpha^4[(1 - \xi') - \lambda(\xi - \xi')] + \beta^4[(1 - \xi') - \lambda(\xi - \xi')] \\ & + 2\alpha^2\beta^2[\xi' + \lambda(\xi - \xi') + \eta' + \lambda(\eta - \eta')]. \end{aligned} \quad (2.13)$$

The equation $\frac{\partial F_{HCM}}{\partial \alpha^2} = 0$ gives the required relationship between the machine parameters ξ, ξ', η , and η' in the form

$$\eta'(1 - \lambda) + \eta\lambda = 1 - 2\xi' - 2\lambda(\xi - \xi'). \quad (2.14)$$

Using (2.14), equation (2.13) reduces to

$$F_{HCM} = (1 - \xi') - \lambda(\xi - \xi'). \quad (2.15)$$

Now the H-S distance between the two mode density operators $\rho_{ab}^{(out)}$ and $\rho_{ab}^{(id)} = \rho_a^{(id)} \otimes \rho_b^{(id)}$ is given by

$$\begin{aligned} D_{ab} &= Tr[\rho_{ab}^{(out)} - \rho_{ab}^{(id)}]^2 \\ &= U_{11}^2 + 2U_{12}^2 + 2U_{13}^2 + U_{22}^2 + 2U_{23}^2 + U_{33}^2, \end{aligned} \quad (2.16)$$

where

$$\begin{aligned}
U_{11} &= \alpha^4 - \alpha^2[\lambda(1 - 2\xi) + (1 - \lambda)(1 - 2\xi')], \\
U_{12} &= U_{21} = \sqrt{2}\alpha^3\beta - \sqrt{2}\alpha\beta(\eta\frac{\lambda}{2} + (1 - \lambda)\frac{\eta'}{2}), \\
U_{13} &= U_{31} = \alpha^2\beta^2, \\
U_{22} &= 2\alpha^2\beta^2 - (2\xi\lambda + 2\xi'(1 - \lambda)), \\
U_{23} &= U_{32} = \sqrt{2}\alpha\beta^3 - \sqrt{2}\alpha\beta(\eta\frac{\lambda}{2} + (1 - \lambda)\frac{\eta'}{2}), \\
U_{33} &= \beta^4 - \beta^2[\lambda(1 - 2\xi) + (1 - \lambda)(1 - 2\xi')].
\end{aligned} \tag{2.17}$$

It is interesting to see that the transformation (2.5) can behave as a state dependent cloner if we relax the condition $\frac{\delta D_{ab}}{\delta \alpha^2} = 0$. Then, it is natural to expect that the machine parameters depend on the input state. Thus, our prime task is to find a relationship between the machine parameters and the input state that minimizes the distortion D_{ab} . We will get an interesting result if we fix any one of the machine parameters ξ or ξ' as $\frac{1}{6}$. Without any loss of generality we can fix $\xi' = \frac{1}{6}$. In doing so, the cloning transformation (2.5) reduces to the combination of B-H optimal universal cloning machine and the B-H type cloning machine.

Now, substituting $\xi' = \frac{1}{6}$ in (2.17) and using (2.14), equation (2.16) can be rewritten as

$$D_{ab} = V_{11}^2 + 2V_{12}^2 + 2V_{13}^2 + V_{22}^2 + 2V_{23}^2 + V_{33}^2, \tag{2.18}$$

where

$$\begin{aligned}
V_{11} &= \alpha^4 - \alpha^2[\lambda(1 - 2\xi) + (1 - \lambda)(\frac{2}{3})], \\
V_{12} &= V_{21} = \sqrt{2}\alpha^3\beta - \sqrt{2}\alpha\beta(\frac{1}{3} - \lambda(\xi - \frac{1}{6})), \\
V_{13} &= V_{31} = \alpha^2\beta^2, \\
V_{22} &= 2\alpha^2\beta^2 - (2\xi\lambda + (\frac{1}{3})(1 - \lambda)), \\
V_{23} &= V_{32} = \sqrt{2}\alpha\beta^3 - \sqrt{2}\alpha\beta(\frac{1}{3} - \lambda(\xi - \frac{1}{6})), \\
V_{33} &= \beta^4 - \beta^2[\lambda(1 - 2\xi) + (1 - \lambda)(\frac{2}{3})].
\end{aligned} \tag{2.19}$$

Now we are in a position to determine the relationship between the machine parameter (ξ) and the input state (α^2) that minimizes the distortion D_{ab} . To obtain the minimum value of D_{ab} for given α and λ , we solve the equation

$$\frac{\delta D_{ab}}{\delta \xi} = 0 \implies \xi = \frac{(9\alpha^2\beta^2 - 2(1 - \lambda))}{12\lambda}, \text{ provided } \lambda \neq 0. \quad (2.20)$$

Now, the cloning machine is defined by those parameters ξ (specified by equation (2.20)) common to the whole family of states that one wants to clone i.e. for given λ , we choose the values of α and β in such a way so that the machine parameter ξ remains invariant. It is clear from equation (2.20) that if we want to minimize D_{ab} then the quantum cloning machine having parameter ξ can be applied on the family of states such that $\alpha^2\beta^2 = \alpha^2(1 - \alpha^2) = \text{constant}$. That means the cloning machine if applied on just four states $|\psi^\pm\rangle_1 = \alpha|0\rangle \pm \beta|1\rangle$, $|\psi^\pm\rangle_2 = \alpha|1\rangle \pm \beta|0\rangle$ will give minimum D_{ab} .

Since the value of the machine parameter ξ cannot be negative, so the parameter λ takes values lying in the interval $[1 - \frac{9\alpha^2(1-\alpha^2)}{2}] < \lambda < 1$.

Also

$$\frac{\delta^2 D_{ab}}{\delta \xi^2} = 16\lambda^2 > 0. \quad (2.21)$$

Therefore, the equation (2.20) represents the required relationship between the machine parameter and the input state which minimizes D_{ab} .

The minimum value of D_{ab} is given by

$$(D_{ab})_{min} = 2\alpha^2\beta^2 - \frac{9\alpha^4\beta^4}{2} \quad (2.22)$$

which depends on α^2 but not on λ .

Substituting $\xi = \frac{9\alpha^2(1-\alpha^2)-2(1-\lambda)}{12\lambda}$ and $\xi' = \frac{1}{6}$ in equation (2.15), we get

$$F_{HCM} = 1 - \frac{3\alpha^2\beta^2}{4}.$$

Table-2.2: Quality of the copies from hybrid cloning machine (B-H state independent transformation + B-H state dependent transformation)

Input state parameter (α^2)	Range of parameter λ	Range of machine parameter $\xi(\lambda)$	$(D_{ab})_{min}$	F_{HCM}
0.1 or 0.9	(0.595, 1.0)	(0.0, 0.0675)	0.14	0.93
0.2 or 0.8	(0.280, 1.0)	(0.0, 0.1200)	0.21	0.88
0.3 or 0.7	(0.055, 1.0)	(0.0, 0.1575)	0.22	0.84
0.4 or 0.6	(0.000, 1.0)	(0.0, 0.1800)	0.22	0.82
0.5	(0.000, 1.0)	(0.0, 0.1875)	0.22	0.81

Illustration of the Table-2.2:

The above table shows that there exists several quantum cloning machines (for different values of ξ) which can clone the four states $\{|\psi^\pm\rangle_1, |\psi^\pm\rangle_2\}$ with the same fidelity. For example, If one of the input state is chosen from the set $S = \{\sqrt{0.1}|0\rangle \pm \sqrt{0.9}|1\rangle, \sqrt{0.9}|0\rangle \pm \sqrt{0.1}|1\rangle\}$ and for a fixed value of λ , say $\lambda = 0.6$ (chosen from the interval (0.595,1.0), then there exist a quantum cloning machine with parameter $\xi = 0.0014$ lying in the interval (0.0, 0.0675), which clone the state from the set S with the fidelity 0.93.

2.2.2 Hybridization of B-H type cloning transformation and phase-covariant quantum cloning transformation

In this subsection, we will show that if B-H type cloning transformation occurs with probability λ and the phase-covariant quantum cloning transformation occurs with probability $1 - \lambda$ then the resulting hybrid quantum cloning machine is a state dependent quantum cloning machine.

The Hybrid cloning transformation is given by

$$\begin{aligned}
|0\rangle|\Sigma\rangle|Q\rangle|n\rangle &\longrightarrow \sqrt{\lambda}[|0\rangle|0\rangle|Q_0\rangle + (|0\rangle|1\rangle + |1\rangle|0\rangle)|Y_0\rangle]|i\rangle \\
&+ (\sqrt{1-\lambda})[(\frac{1}{2} + \frac{1}{\sqrt{8}})|0\rangle|0\rangle + (\frac{1}{2} - \frac{1}{\sqrt{8}})|1\rangle|1\rangle)|\uparrow\rangle + \frac{1}{2}|+\rangle|\downarrow\rangle]|j\rangle,
\end{aligned} \tag{2.23}$$

$$\begin{aligned}
|1\rangle|\Sigma\rangle|Q\rangle|n\rangle &\longrightarrow \sqrt{\lambda}[|1\rangle|1\rangle|Q_1\rangle + (|0\rangle|1\rangle + |1\rangle|0\rangle)|Y_1\rangle]|i\rangle \\
&+ (\sqrt{1-\lambda})[(\frac{1}{2} + \frac{1}{\sqrt{8}})|1\rangle|1\rangle + (\frac{1}{2} - \frac{1}{\sqrt{8}})|0\rangle|0\rangle)|\downarrow\rangle + \frac{1}{2}|+\rangle|\uparrow\rangle]|j\rangle.
\end{aligned} \tag{2.24}$$

When $\lambda = 1$ cloning transformation reduces to B-H type cloning transformation and when $\lambda = 0$ it takes the form of phase-covariant quantum cloning transformation.

The cloning machine (2.23-2.24) approximately copies the information of the input state $|\chi\rangle$ given in subsection 2.2.1 into two identical states described by the same reduced density operator

$$\rho = \lambda[(1 - \xi)|\chi\rangle\langle\chi| + \xi|\bar{\chi}\rangle\langle\bar{\chi}|] + (1 - \lambda)[(\frac{1}{2} + \frac{1}{\sqrt{8}})|\chi\rangle\langle\chi| + (\frac{1}{2} - \frac{1}{\sqrt{8}})|\bar{\chi}\rangle\langle\bar{\chi}|] \quad (2.25)$$

where $|\bar{\chi}\rangle$ is an orthogonal state to $|\chi\rangle$ and ξ is the machine parameter of the B-H type cloning machine given by $\xi = \langle Y_0|Y_0\rangle = \langle Y_1|Y_1\rangle$.

Now, the fidelity is given by

$$F_1 = \langle\chi|\rho|\chi\rangle = (\frac{1}{2} + \frac{1}{\sqrt{8}}) + \lambda(\frac{1}{2} - \frac{1}{\sqrt{8}} - \xi) \quad (2.26)$$

The hybrid quantum cloning machine constructed by combining the B-H type cloning transformation and phase-covariant quantum cloning transformation is state dependent. State dependableness condition arises from the fact that B-H type cloning transformation is state dependent. Consequently, the fidelity F_1 depends on the input state as it depends on the machine parameter $\xi(\alpha^2) = \frac{3\alpha^2(1-\alpha^2)}{4}$. This relationship between the machine parameter ξ associated with the B-H type cloning machine and the input state α^2 is obtained by putting $\lambda = 1$ in equation (2.20).

Following the argument given in previous subsection 2.2.1, we find that the hybrid quantum cloning machine (B-H type cloning transformation + phase covariant quantum cloning transformation) clones the same four states $\{|\psi^\pm\rangle_1, |\psi^\pm\rangle_2\}$ with minimum D_{ab} . Also it can be easily verified that there is no improvement in the quality of cloning of these four states. Therefore, this hybrid quantum cloning machine does not give anything new because it neither involves in cloning of new states nor it gives any improvement in the fidelity of cloning.

2.3 State independent hybrid cloning transformation

In this section, we study one symmetric and two asymmetric universal hybrid quantum cloning machines.

2.3.1 Hybridization of two BH type cloning transformations

In the preceding subsection 2.2.1, we have found that the quantum cloning machine obtained by combining two BH type cloning transformations is state dependent but in this section we will show that a proper combination of two BH type cloning transformations can serve as a state independent cloner also. A hybrid quantum cloning machine (2.5) becomes state independent or universal if the fidelity F_{HCM} defined in equation (2.13) and the deviation D_{ab} defined in equation (2.16), are state independent. From equation (2.15), it is clear that F_{HCM} is state independent. Therefore, the only remaining task is to show the independence of the deviation D_{ab} .

D_{ab} is input state independent if,

$$\begin{aligned} \frac{\partial D_{ab}}{\partial \alpha^2} = 0 &\implies [2(\lambda(1 - 2\xi) + (1 - \lambda)(1 - 2\xi')) - 3]^2 \\ &- [2(\eta\lambda - (1 - \lambda)\eta') - 2]^2 + 8[2\xi\lambda + 2\xi'(1 - \lambda)] - 5 = 0. \end{aligned} \quad (2.27)$$

Using equation (2.14) in equation (2.27), we get

$$\lambda = \frac{(6\xi' - 1)}{6(\xi' - \xi)}, \quad (2.28)$$

provided $\xi \neq \xi'$.

Using the value of λ in (2.15), we get

$$F_{HCM} = \frac{5}{6}. \quad (2.29)$$

If $\xi = \xi' = \frac{1}{6}$, then there is nothing special about the transformation (2.5) because the condition $\xi = \xi' = \frac{1}{6}$, simply takes the transformation (2.5) to B-H state independent quantum cloning transformation. If $\xi \neq \xi'$ and $\xi' \neq \frac{1}{6}$, then the hybrid quantum cloning machine (B-H type transformation + B-H type transformation) will become state independent for all values of ξ and ξ' (provided $\xi \neq \xi'$ and $\xi' \neq \frac{1}{6}$). Therefore the

newly defined hybrid cloning machine generates a class of universal cloning machines for $\lambda = \frac{(6\xi' - 1)}{6(\xi' - \xi)}$ (provided $\xi \neq \xi'$ and $\xi' \neq \frac{1}{6}$). The fidelity of the introduced universal hybrid cloning machine is equal to $\frac{5}{6}$ which is the optimal fidelity one can obtain. Although the machine is universal and optimal for an unknown quantum state but it is different from B-H state independent cloning machine. It is different in the sense that B-H cloning machine is state independent for just only one value of the machine parameter $\xi = \frac{1}{6}$ while the cloning machine defined by (2.5) works as a universal cloner provided λ is given by equation (2.28) and for all values of ξ and ξ' (provided $\xi \neq \xi'$ and $\xi' \neq \frac{1}{6}$).

2.3.2 Hybridization of optimal universal symmetric B-H cloning transformation and optimal universal asymmetric Pauli cloning transformation

An asymmetric quantum cloning machine can be constructed by applying hybridization technique. Using the hybridization procedure we can construct universal asymmetric hybrid quantum cloning machine by combining universal symmetric B-H cloning transformation and optimal universal asymmetric Pauli cloning transformation.

The Hybrid cloning transformation is given by

$$\begin{aligned} |0\rangle|\Sigma\rangle|Q\rangle|n\rangle &\longrightarrow \sqrt{1-\lambda}\left[\sqrt{\frac{2}{3}}|0\rangle|0\rangle|\uparrow\rangle + \sqrt{\frac{1}{6}}(|0\rangle|1\rangle + |1\rangle|0\rangle)|\downarrow\rangle\right]|i\rangle \\ &+ \sqrt{\lambda}\left[\left(\frac{1}{\sqrt{1+p^2+q^2}}\right)(|0\rangle|0\rangle|\uparrow\rangle + (p|0\rangle|1\rangle + q|1\rangle|0\rangle)|\downarrow\rangle)\right]|j\rangle, \end{aligned} \quad (2.30)$$

$$\begin{aligned} |1\rangle|\Sigma\rangle|Q\rangle|n\rangle &\longrightarrow \sqrt{1-\lambda}\left[\sqrt{\frac{2}{3}}|1\rangle|1\rangle|\downarrow\rangle + \sqrt{\frac{1}{6}}(|0\rangle|1\rangle + |1\rangle|0\rangle)|\uparrow\rangle\right]|i\rangle \\ &+ \sqrt{\lambda}\left[\left(\frac{1}{\sqrt{1+p^2+q^2}}\right)(|1\rangle|1\rangle|\downarrow\rangle + (p|1\rangle|0\rangle + q|0\rangle|1\rangle)|\uparrow\rangle)\right]|j\rangle, \end{aligned} \quad (2.31)$$

where $p + q = 1$.

The hybrid cloning machine (2.30-2.31) produces two asymmetric copies of the input state $|\chi\rangle$. These asymmetric cloned states are described by the reduced density operators ρ_1 and ρ_2

$$\rho_1 = \lambda\left[\left(\frac{1}{1+p^2+q^2}\right)((1-q^2+p^2)|\chi\rangle\langle\chi| + q^2I)\right] + (1-\lambda)\left[\frac{5}{6}|\chi\rangle\langle\chi| + \frac{1}{6}|\bar{\chi}\rangle\langle\bar{\chi}|\right], \quad (2.32)$$

$$\rho_2 = \lambda\left[\left(\frac{1}{1+p^2+q^2}\right)((1-p^2+q^2)|\chi\rangle\langle\chi| + p^2I)\right] + (1-\lambda)\left[\frac{5}{6}|\chi\rangle\langle\chi| + \frac{1}{6}|\bar{\chi}\rangle\langle\bar{\chi}|\right]. \quad (2.33)$$

The quality of the asymmetric clones are given by

$$(F_1)_{HCM} = \frac{5}{6} + \left(\frac{\lambda}{2}\right) \left[\frac{(p^2 + 1)}{(p^2 - p + 1)} - \frac{5}{3} \right], \quad (2.34)$$

$$(F_2)_{HCM} = \frac{5}{6} + \left(\frac{\lambda}{2}\right) \left[\frac{(p^2 - 2p + 2)}{(p^2 - p + 1)} - \frac{5}{3} \right]. \quad (2.35)$$

Note: Equations (2.34) and (2.35) show that the hybrid quantum cloning machine tends to B-H state independent quantum cloning machine in the limiting sense when (i) $\lambda \rightarrow 0$ and $0 \leq p \leq 1$ or (ii) $\lambda \rightarrow 1$ and $p = \frac{1}{2}$.

Next we show that the fidelities $(F_1)_{HCM}$ and $(F_2)_{HCM}$ cannot be greater than the optimal value $\frac{5}{6}$ simultaneously i.e. if $(F_1)_{HCM}$ greater than $\frac{5}{6}$ then $(F_2)_{HCM}$ must be less than $\frac{5}{6}$ for all λ 's lying between 0 and 1 and vice-versa.

Without any loss of generality, we assume $(F_1)_{HCM} > \frac{5}{6}$ for $0 < \lambda < 1$. Our task is to find the values of p for which $(F_1)_{HCM} > \frac{5}{6}$.

$$\begin{aligned} (F_1)_{HCM} > \frac{5}{6} &\implies \frac{(p^2 + 1)}{(p^2 - p + 1)} > \frac{5}{3} \\ &\implies (2p - 1)(p - 2) < 0 \\ &\implies (2p - 1) > 0, \text{ Since } p - 2 < 0 \\ &\implies p > \frac{1}{2}. \end{aligned}$$

Now we have to show that if $p > \frac{1}{2}$ then $(F_2)_{HCM} < \frac{5}{6}$. We prove this result by contradiction.

If possible, let $(F_2)_{HCM} > \frac{5}{6}$ for $p > \frac{1}{2}$.

$$\begin{aligned} (F_2)_{HCM} > \frac{5}{6} &\implies \frac{(p^2 - 2p + 2)}{(p^2 - p + 1)} > \frac{5}{3} \\ &\implies (2p - 1)(p + 1) < 0 \\ &\implies (2p - 1) < 0, \text{ Since } p + 1 > 0 \\ &\implies p < \frac{1}{2}. \end{aligned}$$

which contradicts our assumption.

Hence $(F_2)_{HCM} < \frac{5}{6}$ for $p > \frac{1}{2}$. Therefore, we can conclude that the fidelities $(F_1)_{HCM}$ and $(F_2)_{HCM}$ given in (2.34) and (2.35) cannot cross the optimal limit $\frac{5}{6}$ simultaneously.

Furthermore, we construct a table below in which we show the tradeoff between the two fidelities.

Table-2.3: Fidelity of the asymmetric copies produced from asymmetric hybrid quantum cloning machine

p	λ	$(F_1)_{HCM} = \frac{5}{6} + (\frac{\lambda}{2}) \times [\frac{(p^2+1)}{(p^2-p+1)} - \frac{5}{3}]$	$(F_2)_{HCM} = \frac{5}{6} + (\frac{\lambda}{2}) \times [\frac{(p^2-2p+2)}{(p^2-p+1)} - \frac{5}{3}]$	$(F_1)_{HCM} \sim (F_2)_{HCM}$
[0.0,1.0]	0.0	0.83	0.83	0.00 (symmetric copies)
0.0	[0.1,0.9]	[0.80,0.53]	[0.85,0.98]	[0.05,0.45]
0.1	[0.1,0.9]	[0.81,0.58]	[0.85,0.98]	[0.04,0.40]
0.2	[0.1,0.9]	[0.81,0.64]	[0.85,0.96]	[0.04,0.32]
0.3	[0.1,0.9]	[0.82,0.70]	[0.84,0.93]	[0.02,0.23]
0.4	[0.1,0.9]	[0.83,0.77]	[0.84,0.89]	[0.01,0.12]
0.5	[0.1,0.9]	0.83	0.83	0.0 (Symmetric copies)
0.6	[0.1,0.9]	[0.84,0.89]	[0.83,0.77]	[0.01,0.12]
0.7	[0.1,0.9]	[0.84,0.93]	[0.82,0.70]	[0.02,0.23]
0.8	[0.1,0.9]	[0.85,0.96]	[0.81,0.64]	[0.04,0.32]
0.9	[0.1,0.9]	[0.85,0.98]	[0.81,0.58]	[0.04,0.40]
[0.0,1.0]	1.0	$(F_1)_{PCM}$	$(F_2)_{PCM}$	$(F_1)_{PCM} \sim (F_2)_{PCM}$

Illustration of the table 2.3:

For some fixed value of p , we can construct different hybrid quantum cloning machine by combining two independent quantum cloners viz. optimal universal symmetric B-H cloner and optimal universal asymmetric Pauli cloner with different probabilities (λ). These hybrid cloning machine produce two asymmetric copies at the output. In particular, if $p = 0.1$ and $\lambda = 0.2$ then $(F_1)_{HCM} = 0.78$ and $(F_2)_{HCM} = 0.78$. In general, for $p = 0.1$ we find that the quality described by the fidelity $(F_1)_{HCM}$ of one copy decreases from 0.81 to 0.58 while the quality described by the fidelity $(F_2)_{HCM}$ of other copy increases from 0.85 to 0.98 as λ varies from 0.1 to 0.9. As a result the difference between the qualities of the two copies increases from 0.04 to 0.40.

We note that the fidelity of the hybrid quantum cloning machine (B-H cloner + Pauli cloner) depends on the parameter p and λ . From table we can observe that for $p = 0.0$ to $p = 0.4$ one of the output $(F_1)_{HCM}$ behave as a decreasing function and another

output of the asymmetric cloning machine $(F_2)_{HCM}$ behaves as an increasing function. The role of the fidelities $(F_1)_{HCM}$ and $(F_2)_{HCM}$ are swapped for $p = 0.6$ to $p = 0.9$. Here we observe that the asymmetric hybrid cloning machine reduces to B-H symmetric cloning machine in two cases: (i) when $\lambda = 0$ and (ii) when $p = 0.5$. When $\lambda = 1.0$, our asymmetric hybrid cloner also reduces to asymmetric Pauli cloner for all p ($0 \leq p \leq 1$).

2.3.3 Hybridization of universal B-H cloning transformation and universal anti-cloning transformation

In this subsection, we introduce an interesting hybrid quantum-cloning machine, which is a combination of universal B-H cloning machine and a universal anti-cloning machine. The introduced cloning machine is interesting in the sense that it acts like anti-cloning machine which means that the spin direction of the outputs of the cloner are antiparallel. We will show later that the newly introduced Hybrid cloning machine (B-H cloner + Anti-cloner) serves as a better anti-cloner than the existing quantum anti-cloning machine [136]. Also we show that if the values of the machine parameter λ is in the neighborhood of 1 then the values of the two non-identical fidelities lie in the neighborhood of $\frac{5}{6}$

The introduced hybrid anti-cloning transformation is defined by

$$\begin{aligned} |0\rangle|\Sigma\rangle|Q\rangle|n\rangle &\longrightarrow \sqrt{\lambda}\left[\sqrt{\frac{2}{3}}|0\rangle|0\rangle|\uparrow\rangle + \sqrt{\frac{1}{6}}(|0\rangle|1\rangle + |1\rangle|0\rangle)|\downarrow\rangle\right]|i\rangle + (\sqrt{1-\lambda}) \\ &\left[\sqrt{\frac{1}{6}}|0\rangle|0\rangle|\uparrow\rangle + \left(\frac{1}{\sqrt{2}}e^{i\cos^{-1}(\frac{1}{\sqrt{3}})}|0\rangle|1\rangle - \frac{1}{\sqrt{6}}|1\rangle|0\rangle\right)|\rightarrow\rangle + \frac{1}{\sqrt{6}}|1\rangle|1\rangle|\leftarrow\rangle\right]|j\rangle, \end{aligned} \quad (2.36)$$

$$\begin{aligned} |1\rangle|\Sigma\rangle|Q\rangle|n\rangle &\longrightarrow \sqrt{\lambda}\left[\sqrt{\frac{2}{3}}|1\rangle|1\rangle|\downarrow\rangle + \sqrt{\frac{1}{6}}(|0\rangle|1\rangle + |1\rangle|0\rangle)|\uparrow\rangle\right]|i\rangle + (\sqrt{1-\lambda}) \\ &\left[\sqrt{\frac{1}{6}}|1\rangle|1\rangle|\rightarrow\rangle + \left(\frac{1}{\sqrt{2}}e^{i\cos^{-1}(\frac{1}{\sqrt{3}})}|1\rangle|0\rangle - \frac{1}{\sqrt{6}}|0\rangle|1\rangle\right)|\uparrow\rangle + \frac{1}{\sqrt{6}}|0\rangle|0\rangle|\downarrow\rangle\right]|j\rangle, \end{aligned} \quad (2.37)$$

where $|\uparrow\rangle, |\downarrow\rangle, |\rightarrow\rangle, |\leftarrow\rangle$ are mutually orthogonal machine states.

The above defined cloning machine (2.36-2.37) produces two copies which are described

by the reduced density operator in mode ‘a’ and mode ‘b’

$$\begin{aligned} \rho_a = & |0\rangle\langle 0|[\lambda(\frac{5\alpha^2}{6} + \frac{\beta^2}{6}) + (1-\lambda)(\frac{2\alpha^2}{3} + \frac{\beta^2}{3})] + |0\rangle\langle 1|[\lambda\frac{2\alpha\beta}{3} + (1-\lambda)\frac{\alpha\beta}{3}] \\ & + |1\rangle\langle 0|[\lambda\frac{2\alpha\beta}{3} + (1-\lambda)\frac{\alpha\beta}{3}] + |1\rangle\langle 1|[\lambda(\frac{5\beta^2}{6} + \frac{\alpha^2}{6}) + (1-\lambda)(\frac{\alpha^2}{3} + \frac{2\beta^2}{3})], \end{aligned} \quad (2.38)$$

$$\begin{aligned} \rho_b = & |0\rangle\langle 0|[\lambda(\frac{5\alpha^2}{6} + \frac{\beta^2}{6}) + (1-\lambda)(\frac{\alpha^2}{3} + \frac{2\beta^2}{3})] + |0\rangle\langle 1|[\lambda\frac{2\alpha\beta}{3} - (1-\lambda)\frac{\alpha\beta}{3}] \\ & + |1\rangle\langle 0|[\lambda\frac{2\alpha\beta}{3} - (1-\lambda)\frac{\alpha\beta}{3}] + |1\rangle\langle 1|[\lambda(\frac{5\beta^2}{6} + \frac{\alpha^2}{6}) + (1-\lambda)(\frac{2\alpha^2}{3} + \frac{\beta^2}{3})]. \end{aligned} \quad (2.39)$$

Let F_a and F_b denote the fidelities of the two copies with opposite spin direction. These fidelities are given by

$$F_a = \frac{5\lambda}{6} + \frac{2(1-\lambda)}{3}, \quad F_b = \frac{5\lambda}{6} + \frac{(1-\lambda)}{3}. \quad (2.40)$$

It is clear from equation (2.40) that the introduced hybrid anti-cloning machine is asymmetric in nature, i.e., the hybrid quantum cloning machine resulting from Universal B-H cloning machine and universal anti-cloning machine behaves as an asymmetric quantum cloning machine for all values of the parameter λ lying between 0 and 1. The two different fidelities given in (2.40) of the anti-cloning machine can approach the optimal value $\frac{5}{6}$ when the parameter λ approaches one. Here we should note an important fact that both the fidelities tend to $\frac{5}{6}$ but not equal to $\frac{5}{6}$ unless $\lambda = 1$. Hence the fidelities F_a and F_b takes different values in the neighborhood of $\frac{5}{6}$ when the values of λ are lying in the neighborhood of 1. For further illustration we construct a table below:

Table-2.4: Fidelity of the two asymmetric clone produced from hybrid anti-cloning machine

parameter (λ)	$F_a = \frac{5\lambda}{6} + \frac{2(1-\lambda)}{3}$	$F_b = \frac{5\lambda}{6} + \frac{(1-\lambda)}{3}$	Difference between qualities of the two copies $F_a \sim F_b$
0.0	0.67	0.33	0.34
0.1	0.68	0.38	0.30
0.2	0.70	0.43	0.27
0.3	0.72	0.48	0.24
0.4	0.73	0.53	0.20
0.5	0.75	0.58	0.17
0.6	0.77	0.63	0.14
0.7	0.78	0.68	0.10
0.8	0.80	0.73	0.07
0.9	0.82	0.78	0.04
1.0	0.83	0.83	0.00 (Symmetric copies)

Illustration of the Table-2.4:

When the two independent quantum cloner viz. universal B-H quantum copier and universal anti-cloner occurs with probabilities 0.1 and 0.9 respectively in the hybrid cloning machine, it produces two asymmetric copies with fidelity 0.68 and 0.38 respectively.

It is clear that both the fidelities of output copies with opposite spins are increasing function of the parameter λ . Therefore, as λ increases, the values of the fidelities F_a and F_b also increases and approaches towards the optimal cloning fidelity 0.83. The above table shows that when $\lambda = 0$, our hybrid anti-cloner reduces to anti-cloner introduced by Song and Hardy [136]. Also when $\lambda = 1$, we observe that the copies with opposite spin direction change into the copies with same spin direction with optimal fidelity. Therefore, we can conclude that the hybrid anti-cloner performs better than the existing quantum anti-cloning machine.

Chapter 3

Broadcasting of entanglement

The true sign of intelligence is not knowledge but imagination - Albert EINSTEIN

Bell's theorem is easy to understand but hard to believe - Nick Herbert

God [could] vary the laws of Nature, and make worlds of several sorts in several parts of the universe - Isaac Newton

3.1 *Prelude*

Entanglement [56], the heart of quantum information theory plays a crucial role in computational and communicational purposes. As a valuable resource in quantum information processing, quantum entanglement has been widely used in quantum cryptography [55, 135], quantum superdense coding [12] and quantum teleportation [14]. An astonishing feature of quantum information processing is that information can be "encoded" in non-local correlations between two separated particles. The more "pure" is the quantum entanglement, the more "valuable" is the given two-particle state. Therefore, to extract pure quantum entanglement from a partially entangled state, researchers had done lot of works in the past years on purification procedures [15, 47]. In other words, it is possible to compress locally an amount of quantum information. Now generally a question arises: whether the opposite is true or not i.e. can quantum correlations be "decompressed"? This question was tackled by several researchers using the concept of

"Broadcasting of quantum inseparability" [9, 29, 48, 107]. Broadcasting is nothing but a local copying of non-local quantum correlations. Among all the problems regarding entanglement, broadcasting of entanglement is an important issue to consider.

Definition 3.1: Suppose two distant parties A and B share two qubit-entangled state $|s\rangle_{AB} = \alpha|00\rangle_{AB} + \beta|11\rangle_{AB}$, where α is real and β is complex with the condition $\alpha^2 + |\beta|^2 = 1$.

The first qubit belongs to A and the second belongs to B. Each of the two parties now perform local copier on their own qubit and then the input entangled state $|s\rangle$ has been broadcast if for some values of the probability α^2

- (1) non-local output states are inseparable, and
- (2) local output states are separable.

In classical theory one can always broadcast information but in quantum theory, broadcasting is not always possible [10, 93]. H.Barnum et.al. showed that non-commuting mixed states cannot be broadcasted [10]. However for pure states broadcasting is equivalent to cloning. In the process of broadcasting of entanglement, we generally use Peres-Horodecki theorem for showing the inseparability of non-local outputs and separability of local outputs.

V.Buzek, V.Vedral, M.B.Plenio, P.L.Knight and M.Hillery [29] were the first who showed that the decomposition of initial quantum entanglement is possible, i.e. that from a pair of entangled particles, two less entangled pairs can be obtained by local $1 \rightarrow 2$ optimal universal symmetric cloning machine.

When the universal B-H quantum cloners are applied locally on each qubits of the entangled state $|\phi\rangle_{AB} = \alpha|00\rangle_{AB} + \beta|11\rangle_{AB}$, the local output described by the density operator is given by

$$\rho_{AA'} = \rho_{BB'} = \frac{2\alpha^2}{3}|00\rangle\langle 00| + \frac{1}{3}|+\rangle\langle +| + \frac{2\beta^2}{3}|11\rangle\langle 11| \quad (3.1)$$

where $|+\rangle = (\frac{1}{\sqrt{2}})(|01\rangle + |10\rangle)$, A' and B' denote the copies of the input A and B respectively.

while the non-local output described by the density operator is given by

$$\begin{aligned} \rho_{AB'} = \rho_{A'B} = & \frac{(24\alpha^2 + 1)}{36}|00\rangle\langle 00| + \frac{5}{36}(|01\rangle\langle 01| + |10\rangle\langle 10|) + \frac{(24\beta^2 + 1)}{36}|11\rangle\langle 11| + \\ & \frac{4\alpha\beta}{9}(|00\rangle\langle 11| + |11\rangle\langle 00|) \end{aligned} \quad (3.2)$$

From Peres-Horodecki criteria for separability, it follows that $\rho_{AA'}(\rho_{BB'})$ is separable if

$$\frac{1}{2} - \frac{\sqrt{48}}{16} \leq \alpha^2 \leq \frac{1}{2} + \frac{\sqrt{48}}{16} \quad (3.3)$$

and $\rho_{AB'}(\rho_{A'B})$ is inseparable if

$$\frac{1}{2} - \frac{\sqrt{39}}{16} \leq \alpha^2 \leq \frac{1}{2} + \frac{\sqrt{39}}{16} \quad (3.4)$$

Therefore, the entanglement is broadcasted via local state independent quantum cloner if the probability- amplitude-squared α^2 is given by the range

$$\frac{1}{2} - \frac{\sqrt{39}}{16} \leq \alpha^2 \leq \frac{1}{2} + \frac{\sqrt{39}}{16} \quad (3.5)$$

The fidelity of broadcasting is given by

$$F_1(\alpha^2) = \langle \phi | \rho_{AB'} | \phi \rangle = \frac{25}{36} - \frac{4\alpha^2(1 - \alpha^2)}{9} \quad (3.6)$$

It is observed from equation (3.6) that although the state independent cloner is used as a local cloner for broadcasting entanglement, the fidelity of copying an entanglement depends on the input state. Thus, the actions of state independent cloners locally on the entangled state does produce less entangled pairs but its quality depends on the input entangled state.

Hence, the average fidelity is given by

$$\overline{F_1} = \int_0^1 F_1(\alpha^2) d\alpha^2 = \frac{67}{108} \simeq 0.62 \quad (3.7)$$

Further S.Bandyopadhyay and G.Kar [9] studied the broadcasting of entanglement and showed that only those universal quantum cloners whose fidelity is greater than $\frac{1}{2}(1 + \sqrt{\frac{1}{3}})$

are suitable for local copying because only then the non-local output states becomes inseparable for some values of the input parameter α . They proved that an entanglement is optimally broadcast only when optimal quantum cloners are used for local copying and also showed that broadcasting of entanglement into more than two entangled pairs is not possible using only local operations. Later, I.Ghiu [75] investigated the broadcasting of entanglement by using local $1 \rightarrow 2$ optimal universal asymmetric Pauli machines and showed that the inseparability is optimally broadcast when symmetric cloners are applied.

This chapter is divided into two parts:

In the first part, we deal with the problem of how well one can produce two entangled pairs starting from a given entangled pair using state dependent cloner as a local copier. Here we construct a state dependent cloner from B-H quantum cloning transformation by relaxing one of the universality conditions viz. $\frac{\partial D_{ab}}{\partial \alpha^2} = 0$, where $D_{ab} = \text{Tr}[\rho_{ab}^{(out)} - \rho_a^{id} \otimes \rho_b^{id}]^2$. $\rho_{ab}^{(out)}$ describes the entangled output state of the cloner and ρ_a^{id}, ρ_b^{id} describe the input states in modes 'a' and 'b' respectively. Further we show that the length of the interval for probability-amplitude-squared α^2 for broadcasting of entanglement using state dependent cloner can be made larger than the length of the interval for probability-amplitude-squared for broadcasting entanglement using state independent cloner. Moreover, we show that there exists local state dependent cloner which gives better quality copy (in terms of average fidelity) of an entangled pair than the local universal cloner. This part is discussed in details in sections 3.2 and 3.3 of this chapter.

In the second part, we investigate the problem of secretly broadcasting three-qubit entangled state between two distant partners with universal quantum cloning machine and then the result is generalized to generate secret entanglement among three parties. Let us suppose that the two distant partners share an entangled state $|\psi\rangle_{13} = \alpha|00\rangle + \beta|11\rangle$. The two parties then apply optimal universal quantum cloning machine on their respective qubits to produce four qubit state $|\chi\rangle_{1234}$. One party (say, Alice) then performs

measurement on her quantum cloning machine state vectors. After that she informs Bob about her measurement result using Goldenberg and Vaidman's quantum cryptographic scheme based on orthogonal states [83]. Getting measurement result from Alice, other partner (say, Bob) also performs measurement on his quantum cloning machine state vectors and using the same cryptographic scheme, he sends his measurement outcome to Alice. Since the measurement results are interchanged secretly so Alice and Bob share secretly a four qubit state. They again apply the cloning machine on their respective qubits and generate six qubit state $|\phi\rangle_{125346}$. Therefore, both parties have three qubits each. Among six qubit state, we interestingly find that there exists two three qubit state shared by Alice and Bob which are entangled for some values of the input parameter α^2 . Finally, we investigate the problem of secret entanglement broadcasting among three distant parties. To solve this problem, we start with the result of the first part i.e. we assume that the two distant partners (say, Alice and Bob) share a three qubit entangled state. Without any loss of generality, we assume that among three qubits, two are with Alice and one with Bob. Then Alice teleports one of the qubit to the third distant partner (say, Carol). After the completion of the teleportation procedure, we find that the three distant partners share a three qubit entangled state for the same values of the input parameters α^2 as in the first part of the protocol. We discuss this portion in sections 3.4 and 3.5 of this chapter.

This chapter is based on our works "Broadcasting of Inseparability" [3] and "Broadcasting of three-qubit entanglement via local copying and entanglement swapping" [5].

3.2 State dependent B-H quantum cloning machine

In the literature, many state dependent quantum cloners were known. In this section, we also introduce another state dependent cloner. The introduced state dependent cloner is interesting in the sense that it can be constructed from B-H quantum cloning transformation by relaxing one universality condition viz. $\frac{\partial D_{ab}}{\partial \alpha^2} = 0$, where $D_{ab} = \text{Tr}[\rho_{ab}^{(out)} - \rho_a^{id} \otimes \rho_b^{id}]^2$.

The B-H cloning transformation is given by

$$|0\rangle|\Sigma\rangle|Q\rangle \rightarrow |0\rangle|0\rangle|Q_0\rangle + (|0\rangle|1\rangle + |1\rangle|0\rangle)|Y_0\rangle \quad (3.8)$$

$$|1\rangle|\Sigma\rangle|Q\rangle \rightarrow |1\rangle|1\rangle|Q_1\rangle + (|0\rangle|1\rangle + |1\rangle|0\rangle)|Y_1\rangle \quad (3.9)$$

The unitarity of the transformation gives

$$\langle Q_i|Q_i\rangle + 2\langle Y_i|Y_i\rangle = 1, \quad i = 0, 1 \quad (3.10)$$

$$\langle Y_0|Y_1\rangle = \langle Y_1|Y_0\rangle = 0 \quad (3.11)$$

We assume

$$\langle Q_0|Y_0\rangle = \langle Q_1|Y_1\rangle = \langle Q_1|Q_0\rangle = 0 \quad (3.12)$$

Let

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \quad (3.13)$$

with $\alpha^2 + |\beta|^2 = 1$, be the input state. Here we assume α is real and β is complex.

The cloning transformation (3.8-3.9) copies the information of the input state (3.13) partially into two identical states described by the density operators $\rho_a^{(out)}$ and $\rho_b^{(out)}$ respectively.

The reduced density operator $\rho_a^{(out)}$ is given by

$$\begin{aligned} \rho_a^{(out)} &= |0\rangle\langle 0|[\alpha^2 + (|\beta|^2\langle Y_1|Y_1\rangle - \alpha^2\langle Y_0|Y_0\rangle)] + |0\rangle\langle 1|\alpha\beta^*[\langle Q_1|Y_0\rangle + \langle Y_1|Q_0\rangle] + \\ &\quad |1\rangle\langle 0|\alpha\beta[\langle Q_1|Y_0\rangle + \langle Y_1|Q_0\rangle] + |1\rangle\langle 1|[\beta^2 - (|\beta|^2\langle Y_1|Y_1\rangle - \alpha^2\langle Y_0|Y_0\rangle)] \\ &= |0\rangle\langle 0|[\alpha^2 + \lambda(|\beta|^2 - \alpha^2)] + |0\rangle\langle 1|\alpha\beta^*\mu + |1\rangle\langle 0|\alpha\beta\mu + \\ &\quad |1\rangle\langle 1|[\beta^2 - \lambda(|\beta|^2 - \alpha^2)] \end{aligned} \quad (3.14)$$

where

$$\langle Y_0|Y_0\rangle = \langle Y_1|Y_1\rangle = \lambda \quad (3.15)$$

$$\langle Q_0|Y_1\rangle = \langle Q_1|Y_0\rangle = \langle Y_1|Q_0\rangle = \langle Y_0|Q_1\rangle = \frac{\mu}{2} \quad (3.16)$$

The output state described by the density operator $\rho_b^{(out)}$ looks the same as $\rho_a^{(out)}$.

The distortion of the qubit in mode 'a' is

$$D_a = 2\lambda^2(4\alpha^4 - 4\alpha^2 + 1) + 2\alpha^2(1 - \alpha^2)(\mu - 1)^2 \quad (3.17)$$

The distortion D_{ab} is defined by

$$\begin{aligned} D_{ab} &= Tr[\rho_{ab}^{(out)} - \rho_a^{id} \otimes \rho_b^{id}]^2 \\ &= U_{11}^2 + 2|U_{12}|^2 + 2|U_{13}|^2 + U_{22}^2 + 2|U_{23}|^2 + U_{33}^2 \end{aligned} \quad (3.18)$$

where

$$U_{11} = \alpha^4 - \alpha^2(1 - 2\lambda) \quad (3.19)$$

$$U_{12} = \sqrt{2}\alpha^3\beta^* - \frac{\mu}{\sqrt{2}}\alpha\beta^*, U_{21} = (U_{12})^* \quad (3.20)$$

$$U_{13} = \alpha^2(\beta^*)^2, U_{31} = (U_{13})^* \quad (3.21)$$

$$U_{22} = 2\alpha^2|\beta|^2 - 2\lambda \quad (3.22)$$

$$U_{23} = \sqrt{2}\alpha\beta^*|\beta|^2 - \frac{\mu}{\sqrt{2}}\alpha\beta^*, U_{32} = (U_{23})^* \quad (3.23)$$

$$U_{33} = |\beta|^4 - |\beta|^2(1 - 2\lambda) \quad (3.24)$$

The cloning transformation (3.8-3.9) is input state independent if D_a and D_{ab} are input state independent. To make the cloning transformation (3.8-3.9) input state dependent, we assume D_{ab} is input state dependent i.e.

$$\frac{\partial D_{ab}}{\partial \alpha^2} \neq 0 \quad (3.25)$$

The relation between the machine parameters λ and μ is established by solving the equation $\frac{\partial D_a}{\partial \alpha^2} = 0$.

Therefore,

$$\frac{\partial D_a}{\partial \alpha^2} = 0 \implies \mu = 1 - 2\lambda \quad (3.26)$$

The value of the machine parameter λ is restricted from the condition $\frac{\partial D_{ab}}{\partial \alpha^2} \neq 0$. The above condition (3.25) implies that λ can take any value between 0 and $\frac{1}{2}$, except $\frac{1}{6}$. However, if $\lambda = \frac{1}{6}$, then $\frac{\partial D_a}{\partial \alpha^2} = 0$ and $\frac{\partial D_{ab}}{\partial \alpha^2} = 0$, therefore the machine becomes universal

in the sense that D_a and D_{ab} does not depend on the input state.

Putting $\mu = 1 - 2\lambda$ in equation (3.19-3.24) and using equation (3.18), we get

$$\begin{aligned} D_{ab} = & [\alpha^4 - \alpha^2(1 - 2\lambda)]^2 + 4\alpha^2(1 - \alpha^2)(\alpha^2 - \frac{(1 - 2\lambda)}{2})^2 + 2\alpha^4(1 - \alpha^2)^2 + \\ & (2\alpha^2(1 - \alpha^2) - 2\lambda)^2 + 4\alpha^2(1 - \alpha^2)(1 - \alpha^2 - \frac{(1 - 2\lambda)}{2})^2 + \\ & (1 - \alpha^2)^2(2\lambda - \alpha^2)^2 \end{aligned} \quad (3.27)$$

Now, we are in search of the machine parameter ' λ ' for which D_{ab} attains its minimum value.

For maximum or minimum value of D_{ab} , we have

$$\frac{\partial D_{ab}}{\partial \lambda} = 0 \implies \lambda = \frac{3\alpha^2(1 - \alpha^2)}{4} \quad (3.28)$$

Again,

$$\frac{\partial^2 D_{ab}}{\partial \lambda^2} = 16 > 0 \quad (3.29)$$

Thus for the value of λ given in equation (3.28), D_{ab} is minimum.

Table 3.1: Comparison between B-H state dependent and independent cloner.

Input state parameter (α)	For B-H state- dependent cloner		For B-H state- independent cloner	
	Machine parameter $\lambda = \frac{3\alpha^2(1-\alpha^2)}{4}$	Distance between input and output states, $D_a = 2\lambda^2$	Machine parameter $\lambda = \frac{1}{6}$	Distance between input and output states, D_a
0.1	0.007	0.000098	0.167	0.055556
0.2	0.029	0.001682	0.167	0.055556
0.3	0.061	0.007442	0.167	0.055556
0.4	0.101	0.020402	0.167	0.055556
0.5	0.141	0.039762	0.167	0.055556
0.6	0.173	0.059858	0.167	0.055556
0.7	0.187	0.069938	0.167	0.055556
0.8	0.173	0.059858	0.167	0.055556
0.9	0.115	0.026450	0.167	0.055556

Illustration of the table 3.1:

For the input state $(0.1)|0\rangle + \sqrt{0.99}|1\rangle$, we can construct a quantum copying machine with parameter $\lambda = 0.007$ which produces noisy outputs. As a result of copying procedure, the identical copies at the output are distorted

by the amount 0.000098. The quality of the copy of B-H state independent cloner remains same for all input states.

Equation (3.29) shows that D_{ab} has minimum value when the machine parameter λ takes the form given in equation (3.28). Thus we are able to construct a quantum-cloning machine where machine state vectors depend on input state and therefore the quality of the copy depends on the input state i.e. for different input states, machine state vectors take different values and hence the quality of the copy changes.

Putting $\mu = 1 - 2\lambda$ in (3.17), we get $D_a(\alpha^2) = 2\lambda^2$, Since λ depends on α^2 .

3.3 *Broadcasting of entanglement using state dependent B-H quantum cloning machine*

In this section we show that to broadcast an entanglement, state dependent quantum cloning machine is more effective than state independent B-H quantum cloning machine.

Let us consider a general pure entangled state

$$|\chi\rangle_{AB} = \alpha_1|00\rangle + \beta_1|11\rangle + \gamma_1|10\rangle + \delta_1|01\rangle \quad (3.30)$$

where we assume that $\alpha_1, \beta_1, \gamma_1, \delta_1$ are real and satisfy the condition $\alpha_1^2 + \beta_1^2 + \gamma_1^2 + \delta_1^2 = 1$.

The first qubit (A) belongs to Alice and the second qubit (B) belongs to Bob.

The two distant partners Alice and Bob apply their respective state dependent quantum cloner on their qubits to produce two output systems A' and B' respectively. Now to investigate the existence of non-local correlations in two systems described by the non-local density operators $\rho_{AB'}$ and $\rho_{A'B}$, we use Peres-Horodecki criteria. The same criteria is used to test the separability of the local outputs described by the density operators $\rho_{AA'}$ and $\rho_{BB'}$.

The two non-local output states of a copier are described by the density operator $\rho_{AB'}$

and $\rho_{A'B}$,

$$\begin{aligned}\rho_{AB'} = \rho_{A'B} = & C_{11}|00\rangle\langle 00| + C_{44}|11\rangle\langle 11| + C_{22}|01\rangle\langle 01| + C_{33}|10\rangle\langle 10| + \\ & C_{23}(|00\rangle\langle 11| + |11\rangle\langle 00|) + C_{12}(|01\rangle\langle 00| + |00\rangle\langle 01|) + C_{13}(|00\rangle\langle 10| + |10\rangle\langle 00|) + \\ & C_{14}(|01\rangle\langle 10| + |10\rangle\langle 01|) + C_{24}(|01\rangle\langle 11| + |11\rangle\langle 01|) + C_{34}(|11\rangle\langle 10| + \\ & |10\rangle\langle 11|)\end{aligned}\quad (3.31)$$

where

$$C_{11} = \alpha_1^2(1 - \lambda)^2 + \beta_1^2\lambda^2 + \lambda(1 - \lambda)(\delta_1^2 + \gamma_1^2) \quad (3.32)$$

$$C_{12} = \beta_1\gamma_1\lambda\mu + \delta_1\alpha_1\mu(1 - \lambda) \quad (3.33)$$

$$C_{13} = \beta_1\delta_1\lambda\mu + \alpha_1\gamma_1\mu(1 - \lambda) \quad (3.34)$$

$$C_{14} = \mu^2\gamma_1\delta_1 \quad (3.35)$$

$$C_{22} = \delta_1^2(1 - \lambda)^2 + \gamma_1^2\lambda^2 + \lambda(1 - \lambda)(\alpha_1^2 + \beta_1^2) \quad (3.36)$$

$$C_{23} = \mu^2\alpha_1\beta_1 \quad (3.37)$$

$$C_{24} = \alpha_1\gamma_1\lambda\mu + \beta_1\delta_1\mu(1 - \lambda) \quad (3.38)$$

$$C_{33} = \gamma_1^2(1 - \lambda)^2 + \delta_1^2\lambda^2 + \lambda(1 - \lambda)(\alpha_1^2 + \beta_1^2) \quad (3.39)$$

$$C_{34} = \alpha_1\delta_1\lambda\mu + \beta_1\gamma_1\mu(1 - \lambda) \quad (3.40)$$

$$C_{44} = \alpha_1^2\lambda^2 + \beta_1^2(1 - \lambda)^2 + \lambda(1 - \lambda)(\delta_1^2 + \gamma_1^2) \quad (3.41)$$

The two local output states of a copier are described by the density operators $\rho_{AA'}$ and $\rho_{BB'}$,

$$\begin{aligned}\rho_{AA'} = & K_{11}|00\rangle\langle 00| + K_{44}|11\rangle\langle 11| + K_{22}|01\rangle\langle 01| + K_{33}|10\rangle\langle 10| + \\ & K_{12}(|01\rangle\langle 00| + |00\rangle\langle 01|) + K_{13}(|00\rangle\langle 10| + |10\rangle\langle 00|) + \\ & K_{14}(|01\rangle\langle 10| + |10\rangle\langle 01|) + K_{24}(|01\rangle\langle 11| + |11\rangle\langle 01|) + \\ & K_{34}(|11\rangle\langle 10| + |10\rangle\langle 11|)\end{aligned}\quad (3.42)$$

where

$$K_{11} = (1 - 2\lambda)(\alpha_1 + \delta_1)^2 \quad (3.43)$$

$$K_{12} = K_{13} = K_{24} = K_{34} = \left(\frac{\mu}{2}\right)(\alpha_1 + \delta_1)(\beta_1 + \gamma_1) \quad (3.44)$$

$$K_{14} = K_{22} = K_{33} = \lambda + 2\lambda(\alpha_1\delta_1 + \beta_1\gamma_1) \quad (3.45)$$

$$K_{32} = K_{23} = 0, K_{44} = (1 - 2\lambda)(\beta_1 + \gamma_1)^2 \quad (3.46)$$

$$\begin{aligned} \rho_{BB'} = & K'_{11}|00\rangle\langle 00| + K'_{44}|11\rangle\langle 11| + K'_{22}|01\rangle\langle 01| + K'_{33}|10\rangle\langle 10| + \\ & K'_{12}(|01\rangle\langle 00| + |00\rangle\langle 01|) + K'_{13}(|00\rangle\langle 10| + |10\rangle\langle 00|) + \\ & K'_{14}(|01\rangle\langle 10| + |10\rangle\langle 01|) + K'_{24}(|01\rangle\langle 11| + |11\rangle\langle 01|) + \\ & K'_{34}(|11\rangle\langle 10| + |10\rangle\langle 11|) \end{aligned} \quad (3.47)$$

where

$$K'_{11} = (1 - 2\lambda)(\alpha_1 + \gamma_1)^2 \quad (3.48)$$

$$K'_{12} = K'_{13} = K'_{24} = K'_{34} = \left(\frac{\mu}{2}\right)(\alpha_1 + \gamma_1)(\beta_1 + \delta_1) \quad (3.49)$$

$$K'_{14} = K'_{22} = K'_{33} = \lambda + 2\lambda(\alpha_1\gamma_1 + \beta_1\delta_1) \quad (3.50)$$

$$K'_{32} = K'_{23} = 0, K'_{44} = (1 - 2\lambda)(\beta_1 + \delta_1)^2 \quad (3.51)$$

The composite systems described by the density operators $\rho_{AB'}$ and $\rho_{A'B}$ are inseparable if at least one of the determinants W_3 and W_4 is negative and W_2 is non-negative, where

$$\begin{aligned} W_3 = & \begin{vmatrix} C_{11} & C_{12} & C_{13} \\ C_{12} & C_{22} & C_{23} \\ C_{13} & C_{23} & C_{33} \end{vmatrix}, \quad W_4 = \begin{vmatrix} C_{11} & C_{12} & C_{13} & C_{14} \\ C_{12} & C_{22} & C_{23} & C_{24} \\ C_{13} & C_{23} & C_{33} & C_{34} \\ C_{14} & C_{24} & C_{34} & C_{44} \end{vmatrix}, \\ W_2 = & \begin{vmatrix} C_{11} & C_{12} \\ C_{12} & C_{22} \end{vmatrix} \end{aligned} \quad (3.52)$$

The entries in the determinants are given by the equations (3.32-3.41).

The local output state in Alice's Hilbert space described by the density operator $\rho_{AA'}$ is

separable if

$$W_3 = \begin{vmatrix} K_{11} & K_{12} & K_{13} \\ K_{12} & K_{22} & K_{23} \\ K_{13} & K_{23} & K_{33} \end{vmatrix} \geq 0, \quad W_4 = \begin{vmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ K_{12} & K_{22} & K_{23} & K_{24} \\ K_{13} & K_{23} & K_{33} & K_{34} \\ K_{14} & K_{24} & K_{34} & K_{44} \end{vmatrix} \geq 0, \\ W_2 = \begin{vmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{vmatrix} \geq 0 \quad (3.53)$$

The entries in the determinants are given by the equations (3.43-3.46).

The local output state in Bob's Hilbert space described by the density operator $\rho_{BB'}$ is separable if

$$W_3 = \begin{vmatrix} K'_{11} & K'_{12} & K'_{13} \\ K'_{12} & K'_{22} & K'_{23} \\ K'_{13} & K'_{23} & K'_{33} \end{vmatrix} \geq 0, \quad W_4 = \begin{vmatrix} K'_{11} & K'_{12} & K'_{13} & K'_{14} \\ K'_{12} & K'_{22} & K'_{23} & K'_{24} \\ K'_{13} & K'_{23} & K'_{33} & K'_{34} \\ K'_{14} & K'_{24} & K'_{34} & K'_{44} \end{vmatrix} \geq 0, \\ W_2 = \begin{vmatrix} K'_{11} & K'_{12} \\ K'_{12} & K'_{22} \end{vmatrix} \geq 0 \quad (3.54)$$

The entries in the determinants are given by the equations (3.48-3.51).

The general pure entangled state (3.30) can be broadcast if the equations (3.52)-(3.54) are satisfied.

For simplicity and without any loss of generality, we assume that the two distant parties Alice and Bob share a pair of particles prepared in the pure entangled state

$$|\chi\rangle = \alpha_1|00\rangle_{AB} + \beta_1|11\rangle_{AB} \quad (3.55)$$

where α_1 is real and β_1 is a complex number such that $\alpha_1^2 + |\beta_1|^2 = 1$.

Alice and Bob then apply the state dependent quantum cloner as a local copier on their qubits. As a result, the two non-local output states described by the density operators $\rho_{AB'}$ and $\rho_{A'B}$ are given by

$$\begin{aligned} \rho_{AB'} = \rho_{A'B} = & |00\rangle\langle 00|[\alpha_1^2(1-2\lambda) + \lambda^2] + \lambda(1-\lambda)(|01\rangle\langle 01| + |10\rangle\langle 10|) + \\ & |11\rangle\langle 11|(|\beta_1|^2(1-2\lambda) + \lambda^2) + \alpha_1\beta_1^*\mu^2|00\rangle\langle 11| + \\ & \alpha_1\beta_1\mu^2|11\rangle\langle 00| \end{aligned} \quad (3.56)$$

It follows from the Peres-Horodecki theorem that $\rho_{AB'}$ and $\rho_{A'B}$ are inseparable if

$$W_4 = \begin{vmatrix} (1-2\lambda)\alpha_1^2 & 0 & 0 & 0 \\ 0 & \lambda(1-\lambda) & \alpha_1\beta_1^*\mu^2 & 0 \\ 0 & \alpha_1\beta_1\mu^2 & \lambda(1-\lambda) & 0 \\ 0 & 0 & 0 & |\beta_1|^2(1-2\lambda) + \lambda^2 \end{vmatrix} < 0$$

$$\Rightarrow \alpha_1^4\mu^4 - \alpha_1^2\mu^4 + \lambda^2(1-\lambda)^2 < 0$$

$$\Rightarrow \frac{1}{2} - \left(\frac{\sqrt{\mu^4 - 4\lambda^2(1-\lambda)^2}}{2\mu^2} \right) < \alpha_1^2 < \frac{1}{2} + \left(\frac{\sqrt{\mu^4 - 4\lambda^2(1-\lambda)^2}}{2\mu^2} \right)$$

$$\Rightarrow \frac{1}{2} - \left(\frac{\sqrt{(1-2\lambda)^4 - 4\lambda^2(1-\lambda)^2}}{2(1-2\lambda)^2} \right) < \alpha_1^2 < \frac{1}{2} + \left(\frac{\sqrt{(1-2\lambda)^4 - 4\lambda^2(1-\lambda)^2}}{2(1-2\lambda)^2} \right)$$

Also we note that $W_3 < 0$ and $W_2 \geq 0$.

The local density operators $\rho_{AA'}$ and $\rho_{BB'}$ are given by

$$\begin{aligned} \rho_{AA'} = \rho_{BB'} &= |00\rangle\langle 00|\alpha_1^2(1-2\lambda) + \lambda(|01\rangle\langle 01| + |10\rangle\langle 10| + |01\rangle\langle 10| + |10\rangle\langle 01|) + \\ &|11\rangle\langle 11||\beta_1|^2(1-2\lambda) \end{aligned} \quad (3.57)$$

Now $\rho_{AA'}$ and $\rho_{BB'}$ are separable if $W_2 \geq 0$, $W_3 \geq 0$ and $W_4 \geq 0$.

$$W_4 = \begin{vmatrix} (1-2\lambda)\alpha_1^2 & 0 & 0 & \lambda \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ \lambda & 0 & 0 & |\beta_1|^2(1-2\lambda) \end{vmatrix} \geq 0$$

$$\Rightarrow \alpha_1^4(1-2\lambda)^2 - \alpha_1^2(1-2\lambda)^2 + \lambda^2 \leq 0$$

$$\Rightarrow \frac{1}{2} - \frac{\sqrt{1-4\lambda}}{2(1-2\lambda)} \leq \alpha_1^2 \leq \frac{1}{2} + \frac{\sqrt{1-4\lambda}}{2(1-2\lambda)}$$

Table 3.2: Intervals representing the separability and inseparability between two systems.

Machine parameter, λ	Interval(I_1) for inseparability between systems $(A - B')$ and $(A' - B)$	Interval(I_2) for separability between systems $(A - A')$ and $(B - B')$	Common interval between (I_1) and (I_2)
0.007	(0.00005, 0.99994)	(0.00005, 0.99994)	(0.00005, 0.99994)
0.029	(0.00101, 0.99899)	(0.00094, 0.99905)	(0.00101, 0.99899)
0.061	(0.00555, 0.99444)	(0.00485, 0.99514)	(0.00555, 0.99444)
0.101	(0.02076, 0.97923)	(0.01628, 0.98371)	(0.02076, 0.97923)
0.115	(0.03038, 0.96961)	(0.02282, 0.97717)	(0.03038, 0.96961)
0.141	(0.05863, 0.94136)	(0.04017, 0.95982)	(0.05863, 0.94136)
0.159	(0.09091, 0.90908)	(0.05768, 0.94231)	(0.09091, 0.90908)
0.173	(0.12836, 0.87163)	(0.07570, 0.92429)	(0.12836, 0.87163)
0.187	(0.18458, 0.81541)	(0.09904, 0.90095)	(0.18458, 0.81541)

Illustration of the table 3.2:

If locally the quantum cloning machine with parameter $\lambda = 0.029$ is used for broadcasting an entanglement, then the 2-qubit systems $(A - B')$ and $(A' - B)$ possess non-local properties for $0.00101 < \alpha_1^2 < 0.99899$ and the systems $(A - A')$ and $(B - B')$ possess local properties for $0.00094 < \alpha_1^2 < 0.99905$. Therefore, the class of input entangled states with parameter α_1^2 lying in the interval $(0.00101, 0.99899)$ has been broadcast when the local copying machine with parameter $\lambda = 0.029$ is used.

We can observe from the above table that in the last two cases, the length of the intervals for broadcasting via state dependent cloner are smaller than the length of the interval for broadcasting discussed by Buzek et.al. while the situation is opposite in the remaining cases.

Now to see how well the local state dependent quantum cloners produce two entangled pairs from a single pair, we have to calculate the amount of overlapping between the input entangled state and the output entangled state described by the density operator $\rho_{AB'}(\rho_{A'B})$.

The fidelity of broadcasting of inseparability is given by

$$F(\alpha_1^2) = \langle \chi | \rho_{AB'} | \chi \rangle = (1 - \lambda)^2 - 4\alpha_1^2(1 - \alpha_1^2)\lambda(1 - 2\lambda) \quad (3.58)$$

Table 3.3: Quality of the copies of an entangled state produced from state dependent B-H quantum cloning machine

Amplitude (α)	Machine parameter $\lambda = \frac{3\alpha^2(1-\alpha^2)}{4}$	$F(\alpha_1^2) = (1 - \lambda)^2 + 4\lambda(1 - 2\lambda)\alpha_1^2(1 - \alpha_1^2)$
0.1	0.007	0.99
0.2	0.029	0.94
0.3	0.061	0.86
0.4	0.101	0.76
0.5	0.141	0.66
0.6	0.173	0.58
0.7	0.187	0.54
0.8	0.173	0.58
0.9	0.115	0.72

Illustration of the table 3.3:

The state dependent B-H quantum cloning machine with parameter $\lambda = 0.007$ produce two less entangled copies of the input entangled state $(0.1)|0\rangle + \sqrt{0.99}|1\rangle$ with fidelity 0.99.

3.4 Secret broadcasting of 3-qubit entangled state between two distant partners

All the previous works on the broadcasting of entanglement dealt with the generation of two 2-qubit entangled state starting from a 2-qubit entangled state using either optimal universal symmetric cloner or asymmetric cloner. The generated two qubit entangled state can be used as a quantum channel in quantum cryptography, quantum teleportation etc. The advantage of our protocol over other protocols of broadcasting is that we are able to provide a protocol which generates secret quantum channel between distant partners. The introduced protocol generates two 3-qubit entangled states between two distant partners starting from a 2-qubit entangled state and also provides the security of the generated quantum channel. Not only that we also generalize our protocol from two parties to three parties and show that the generated 3-qubit entangled states can serve as a secured quantum channel between three distant parties.

Few definitions

Let the shared entangled state described by the two qubit density operator be ρ_{13} . Using B-H quantum cloning machine twice by the distant partners (Alice and Bob) on their respective qubits, they generate total six-qubit state ρ_{125346} between them. Therefore, Alice has three qubits '1', '2' and '5' and Bob possesses three qubits '3', '4' and '6'.

Definition 3.2: The three-qubit entanglement is said to be broadcast if (i) Any of the two local outputs (say (ρ_{12}, ρ_{15}) in Alice's side and (ρ_{34}, ρ_{36}) in Bob's side) are separable (ii) One local output (say ρ_{25} in Alice's side and ρ_{46} in Bob's side) is inseparable and associated with these local inseparable output, two non-local outputs (say (ρ_{23}, ρ_{35}) and (ρ_{14}, ρ_{16})) are inseparable.

Definition 3.3: An entanglement is said to be closed if each party has non-local correlation with other parties. For instance, any three particle entangled state described by the density operator ρ_{325} is closed if ρ_{32}, ρ_{25} and ρ_{35} are entangled state. Otherwise, it is said to be an open entanglement (See figure 3.7).

3.5 Discussion of quantum cryptographic scheme based on orthogonal states

Since non-orthogonal states cannot be cloned so many quantum cryptographic protocols were designed on the basis of No-cloning principle which uses non-orthogonal states as the carriers of information. In 1995, L. Goldenberg and L. Vaidman introduced a quantum cryptographic scheme which was based on orthogonal states [83]. The two distant partners Alice and Bob uses Goldenberg and Vaidman's quantum cryptographic scheme to send their measurement results. Now we describe Goldenberg and Vaidman's quantum cryptographic scheme adopted by Alice to send her measurement result to Bob.

Without any loss of generality, let Alice consider the same experimental setup as used in [83] to send her measurement result to Bob. The set up consists of a Mach-Zehnder interferometer with two storage rings, SR_1 and SR_2 , of equal time delays. The set up is described in figure-3.1. Alice can transmit a bit by sending a single particle either from the source $S_{|1\rangle^A}$ (sending 0) or from the source $S_{|1\rangle^A}$ (sending 1), where

$$|\uparrow\rangle^A = \frac{1}{\sqrt{2}}(|a\rangle + |b\rangle) \quad (3.59)$$

$$|\downarrow\rangle^A = \frac{1}{\sqrt{2}}(|a\rangle - |b\rangle) \quad (3.60)$$

Alice then registered the sending time t_s for later use. The particle passes through the first beam-splitter BS_1 and evolves into a superposition of two localized wavepackets: $|a\rangle$, moving in the upper channel and $|b\rangle$, moving in the bottom channel. The wavepacket $|b\rangle$ is delayed for some time τ in the storage ring SR_1 while $|a\rangle$ is moving in the upper channel. When $|a\rangle$ arrives to the storage ring SR_2 at Bob's site, wavepacket $|b\rangle$ starts moving on the bottom channel towards Bob. Let us assume for simplicity that the travelling time of the particles from Alice to Bob be ϵ smaller than the delayed time τ . During the flight-time of $|b\rangle$, wavepacket $|a\rangle$ is delayed in SR_2 . Finally, the two wavepackets arrive simultaneously to the second beam-splitter BS_2 and interfere. A particle started in the state $|\uparrow\rangle^A$ emerges at the detector $D_{|\uparrow\rangle}$, and a particle started in the state $|\downarrow\rangle^A$ emerges at the detector $D_{|\downarrow\rangle}$. Bob, detecting the arriving particle, receives the bit sent by Alice: $D_{|\uparrow\rangle}$ activated means '0' and $D_{|\downarrow\rangle}$ activated means '1'. As soon as Bob detects the arriving particle, he registers the receiving time of the particle t_r . Now the only task remaining for Alice and Bob is to detect the third party 'Eve'. To do the same, Alice and Bob perform two tests (using a classical channel). First, they compare the sending time t_s with the receiving time t_r for each particle, where $t_r = t_s + \tau + \epsilon$. Second, they look for changes in the data by comparing a portion of the transmitted bits with the same portion of the received bits. If, for any checked bit, the timing is not respected or anti-correlated bits are found, the users learn about the intervention of Eve. In this way, Alice sends her measurement result secretly (either $|\uparrow\rangle^A$ or $|\downarrow\rangle^A$) to Bob.

If Eve wants to learn the message in a mid-way sending from Alice to Bob and at the same time if she wants to omit her presence in the running protocol then she has to obey the two basic requirements: 1) since the data is encoded in the relative phase between the two wavepackets $|a\rangle$ and $|b\rangle$ so she has to do something which makes the phase same

at t_s and t_r . 2) She have to keep in mind that the two wavepackets must arrive together to BS_2 at the correct time, otherwise a timing problem occurs.

Description of protocol

Let us consider that the two qubit entangled state $|\psi\rangle_{13}$ is shared between two distant partners popularly known as Alice and Bob. The particles '1' possessed by Alice and the paticle '3' possessed by Bob respectively. Alice and Bob then operate quantum cloning machines on their respective qubits. After cloning procedure, Alice performs measurement on the quantum cloning machine state vector and sends the measurement result to Bob using the Goldenberg and Vaidman's quantum cryptographic scheme based [83] on orthogonal states. After getting measurement result from Alice; Bob performs measurement on his quantum cloning machine state vector and sends the measurement result to Alice using the same cryptographic scheme adopted by Alice. Consequently, the two distant partners share a four qubit state $|\zeta\rangle_{1234}$. Now Alice has two qubits '1' and '2' and Bob '3' and '4' respectively. Both of them again operates quantum cloning machine on one of the qubits, they holds. As a result, the distant parties now share six qubit state $|\phi\rangle_{125346}$ in which three qubits '1','2' and '5' possessed by Alice and remaining three qubits '3','4' and '6' possessed by Bob. Here we show that there exist two 3-qubit entangled states between two distant partners for some values of the input parameter α^2 and therefore it is possible to show that secret broadcasting of 3-qubit entangled state using only universal quantum cloning machine. The word 'secret' is justified by observing an important fact that the transmission of measurement results from Alice to Bob and Bob to Alice have been done by using Goldenberg and Vaidman's quantum cryptographic scheme. Therefore, message regarding measurement results have been transmitted secretly between two distant partners. Hence, the broadcasted three-qubit entangled state is only known to Alice and Bob and not to the third party 'Eve'. As a result, these newly generated three-qubit entangled states can be used as secret quantum channels in various quantum cryptographic scheme. Also, it is very difficult to hack the

quantum information flowing between two distant partners via our proposed quantum channel because to hack the quantum information the hacker (say, Eve) has to do two things: First, she has to gather knowledge about the initially shared entangled state and secondly, she has to collect information about the measurement result performed by two distant partners. Therefore, the quantum channel generated by our protocol is more secured and hence can be used in various protocols viz. quantum key distribution protocols [34, 105] etc.

Now to understand our protocol more clearly, we again discuss the whole protocol below by considering a specific example.

Step -1

Let the two particle entangled state shared by two distant partners Alice and Bob be given by

$$|\psi\rangle_{13} = \alpha|00\rangle + \beta|11\rangle \quad (3.61)$$

where α is real and β is complex with $\alpha^2 + |\beta|^2 = 1$. (See figure-3.2).

Step-2

The B-H quantum copier is given by the transformation

$$|0\rangle|\Sigma\rangle|Q\rangle \rightarrow \sqrt{\frac{2}{3}}|00\rangle|Q_0\rangle + \frac{1}{\sqrt{3}}|\psi^+\rangle|Q_1\rangle \quad (3.62)$$

$$|1\rangle|\Sigma\rangle|Q\rangle \rightarrow \sqrt{\frac{2}{3}}|11\rangle|Q_1\rangle + \frac{1}{\sqrt{3}}|\psi^+\rangle|Q_0\rangle \quad (3.63)$$

where $|\psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$ and $|Q_0\rangle, |Q_1\rangle$ are orthogonal quantum cloning machine state vectors. Alice and Bob then operates B-H quantum cloning machine locally to copy the state of their respective particles. Therefore, the combined system of four qubits is given by

$$\begin{aligned} |\chi\rangle_{1234} = & [(\frac{2\alpha}{3}|0000\rangle + \frac{\beta}{3}|\psi^+\rangle|\psi^+\rangle)|Q_0\rangle^B + (\frac{\sqrt{2}\alpha}{3}|00\rangle|\psi^+\rangle + \frac{\sqrt{2}\beta}{3}|\psi^+\rangle|11\rangle)|Q_1\rangle^B] \\ & |Q_0\rangle^A + [(\frac{\sqrt{2}\alpha}{3}|\psi^+\rangle|00\rangle + \frac{\sqrt{2}\beta}{3}|11\rangle|\psi^+\rangle)|Q_0\rangle^B + (\frac{\alpha}{3}|\psi^+\rangle|\psi^+\rangle + \\ & \frac{2\beta}{3}|1111\rangle)|Q_1\rangle^B]|Q_1\rangle^A \end{aligned} \quad (3.64)$$

The subscripts 1,2 and 3,4 refer to two approximate copy qubits in the Alice's and Bob's Hilbert space respectively. Also $|\rangle^A$ and $|\rangle^B$ denote quantum cloning machine state vectors in Alice's and Bob's Hilbert space respectively. (See figure-3.3).

Alice then performs measurement on the quantum cloning machine state vectors in the basis $\{|Q_0\rangle^A, |Q_1\rangle^A\}$. Thereafter Alice informs Bob about her measurement result using Goldenberg and Vaidman's [83] quantum cryptographic scheme based on orthogonal states. After getting measurement result from Alice, Bob also performs measurement on the quantum cloning machine state vectors in the basis $\{|Q_0\rangle^B, |Q_1\rangle^B\}$ and then using the same cryptographic scheme, he sends his measurement outcome to Alice. In this way Alice and Bob interchange their measurement results secretly.

Step-3

After measurement, let the state shared by Alice and Bob be given by

$$|\zeta_a\rangle_{1234} = \frac{1}{\sqrt{N}} \left[\frac{2\alpha}{3} |0000\rangle + \frac{\beta}{3} |\psi^+\rangle |\psi^+\rangle \right] \quad (3.65)$$

Where $N = \frac{3\alpha^2+1}{9}$ represents the normalization factor.

Afterward, Alice and Bob again operate their respective cloners on the qubits '2' and '4' respectively and therefore, the total state of six qubits is given by

$$\begin{aligned} |\phi\rangle_{125346} = & \frac{1}{\sqrt{N}} \left[\frac{2\alpha}{3} [|0\rangle_1 \otimes (\sqrt{\frac{2}{3}} |00\rangle |Q_0\rangle + \frac{1}{\sqrt{3}} |\psi^+\rangle |Q_1\rangle)_{25} \otimes |0\rangle_3 \otimes (\sqrt{\frac{2}{3}} |00\rangle |Q_0\rangle + \right. \\ & \frac{1}{\sqrt{3}} |\psi^+\rangle |Q_1\rangle)_{46} + \frac{\beta}{6} [|0\rangle_1 \otimes (\sqrt{\frac{2}{3}} |11\rangle |Q_1\rangle + \frac{1}{\sqrt{3}} |\psi^+\rangle |Q_0\rangle)_{25} \otimes |0\rangle_3 \otimes \\ & (\sqrt{\frac{2}{3}} |11\rangle |Q_1\rangle + \frac{1}{\sqrt{3}} |\psi^+\rangle |Q_0\rangle)_{46} + |0\rangle_1 \otimes (\sqrt{\frac{2}{3}} |11\rangle |Q_1\rangle + \frac{1}{\sqrt{3}} |\psi^+\rangle |Q_0\rangle)_{25} \\ & \otimes |1\rangle_3 \otimes (\sqrt{\frac{2}{3}} |00\rangle |Q_0\rangle + \frac{1}{\sqrt{3}} |\psi^+\rangle |Q_1\rangle)_{46} + |1\rangle_1 \otimes (\sqrt{\frac{2}{3}} |00\rangle |Q_0\rangle + \\ & \frac{1}{\sqrt{3}} |\psi^+\rangle |Q_1\rangle)_{25} \otimes |0\rangle_3 \otimes (\sqrt{\frac{2}{3}} |11\rangle |Q_1\rangle + \frac{1}{\sqrt{3}} |\psi^+\rangle |Q_0\rangle)_{46} + |1\rangle_1 \otimes \\ & (\sqrt{\frac{2}{3}} |00\rangle |Q_0\rangle + \frac{1}{\sqrt{3}} |\psi^+\rangle |Q_1\rangle)_{25} \otimes |1\rangle_3 \otimes (\sqrt{\frac{2}{3}} |00\rangle |Q_0\rangle + \\ & \left. \frac{1}{\sqrt{3}} |\psi^+\rangle |Q_1\rangle)_{46} \right] \quad (3.66) \end{aligned}$$

Now our task is to see whether we can generate two 3-qubit entangled state from above six qubit state or not. To examine the above fact, we have to consider two 3-qubit states

described by the density operators ρ_{146} and ρ_{325} (See figure-3.4).

The density operator ρ_{146} is given by

$$\begin{aligned} \rho_{146} = & \frac{1}{N} \left[\frac{4\alpha^2}{9} \left(\frac{2}{3} |000\rangle\langle 000| + \frac{1}{3} |0\psi^+\rangle\langle 0\psi^+| \right) + \frac{\alpha\beta^*}{9} \left(\frac{\sqrt{2}}{3} |000\rangle\langle 1\psi^+| + \frac{\sqrt{2}}{3} |0\psi^+\rangle\langle 111| \right) + \right. \\ & \frac{\alpha\beta}{9} \left(\frac{\sqrt{2}}{3} |111\rangle\langle 0\psi^+| + \frac{\sqrt{2}}{3} |1\psi^+\rangle\langle 000| \right) + \frac{|\beta|^2}{36} \left(\frac{2}{3} |011\rangle\langle 011| + \frac{2}{3} |0\psi^+\rangle\langle 0\psi^+| + \right. \\ & \left. \left. \frac{2}{3} |000\rangle\langle 000| + \frac{2}{3} |111\rangle\langle 111| + \frac{2}{3} |1\psi^+\rangle\langle 1\psi^+| + \frac{2}{3} |100\rangle\langle 100| \right) \right] \end{aligned} \quad (3.67)$$

The density operator ρ_{325} describes the other 3-qubit state looks exactly the same as ρ_{146} .

Now to show the state described by the density operator ρ_{146} is entangled, we have to show that the two qubit states described by the density operators ρ_{14}, ρ_{16} and ρ_{46} are entangled i.e. we have to show that there exist some values of the input state parameter α^2 for which the three-qubit state is a closed entangled state.

The reduced density operators ρ_{14}, ρ_{16} and ρ_{46} are given by

$$\begin{aligned} \rho_{16} = \rho_{14} = & \frac{1}{N} \left[\frac{4\alpha^2}{9} \left(\frac{5}{6} |00\rangle\langle 00| + \frac{1}{6} |01\rangle\langle 01| \right) + \frac{2\alpha\beta^*}{27} |00\rangle\langle 11| + \frac{2\alpha\beta}{27} |11\rangle\langle 00| + \right. \\ & \left. \frac{|\beta|^2}{36} (|00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10| + |11\rangle\langle 11|) \right] \end{aligned} \quad (3.68)$$

$$\begin{aligned} \rho_{46} = & \frac{1}{N} \left[\frac{4\alpha^2}{9} \left(\frac{2}{3} |00\rangle\langle 00| + \frac{1}{6} (|01\rangle\langle 01| + |01\rangle\langle 10| + |10\rangle\langle 01| + |10\rangle\langle 10|) \right) + \right. \\ & \frac{|\beta|^2}{36} \left(\frac{4}{3} |00\rangle\langle 00| + \frac{4}{3} |11\rangle\langle 11| + \frac{2}{3} (|01\rangle\langle 01| + |01\rangle\langle 10| + |10\rangle\langle 01| + \right. \\ & \left. \left. |10\rangle\langle 10|) \right) \right] \end{aligned} \quad (3.69)$$

Using Peres-Horodecki theorem, we find that the states described by the density operators ρ_{16} and ρ_{14} are entangled if $0.18 < \alpha^2 < 1$ and the state described by the density operator ρ_{46} is entangled if $0.61 < \alpha^2 < 1$. Therefore, we can say that the state described by the density operator ρ_{146} is a closed three qubit entangled state if $0.61 < \alpha^2 < 1$. Similarly, the other reduced density operator ρ_{325} describe a closed entangled state if $0.61 < \alpha^2 < 1$.

Also the other two-qubit states described by the density operators $\rho_{12}, \rho_{15}, \rho_{34}$ and ρ_{36}

are given by

$$\begin{aligned} \rho_{12} = \rho_{15} = \rho_{34} = \rho_{36} = & \frac{1}{N} \left[\frac{4\alpha^2}{9} \left(\frac{5}{6} |00\rangle\langle 00| + \frac{1}{6} |01\rangle\langle 01| \right) + \frac{|\beta|^2}{36} \left(\frac{1}{3} |00\rangle\langle 00| + \frac{5}{3} |01\rangle\langle 01| + \right. \right. \\ & \left. \left. \frac{4}{3} |01\rangle\langle 10| + \frac{4}{3} |10\rangle\langle 01| + \frac{5}{3} |10\rangle\langle 10| + \frac{1}{3} |11\rangle\langle 11| \right) \right] \end{aligned} \quad (3.70)$$

These density operators are separable only when $0.27 < \alpha^2 < 1$. Hence, broadcasting of three-qubit entangled state is possible when $0.61 < \alpha^2 < 1$.

Now, our task is to find out how is the entanglement distributed over the state i.e. how much are the two qubit density operators ρ_{16} , ρ_{14} and ρ_{46} are entangled. To evaluate the amount of entanglement, we have to calculate the concurrence defined by Wootters [148] and hence entanglement of formation.

After some tedious calculations, we find that the concurrence and hence the entanglement of formation depends on the probability α^2 . Therefore, we have to calculate the amount of entanglement in the 2-qubit states described by the reduced density operators ρ_{16} , ρ_{14} and ρ_{46} in the range $0.61 < \alpha^2 < 1$ because the two qubit reduced density operators are entangled in this range of the input state parameter α^2 . Since concurrence depends on α^2 so it varies as α^2 varies. Therefore, when $0.61 < \alpha^2 < 1$, the concurrences for the mixed states described by density operators ρ_{16}, ρ_{14} varies from 0.17 to 0.29 while the concurrence for the mixed states described by density operators ρ_{46} varies from 0.08 to 0.15 respectively. Using equations (1.9) and (1.10), we find that the amount of entanglement between the states described by the density operators ρ_{16}, ρ_{14} varies from 0.06 to 0.15 while the amount of entanglement between the states described by the density operator ρ_{46} varies from 0.01 to 0.03 respectively. Therefore, the generated three-qubit entangled state is a weak closed entangled state in the sense that the amount of entanglement in the two-qubit density operators are very low. Further, the above results show that the entanglement between the qubits 1 and 6 (1 and 4) is higher than that between the the qubits 4 and 6.

Furthermore, if the measurement results are $\frac{\sqrt{2}\alpha}{3}|00\rangle|\psi^+\rangle + \frac{\sqrt{2}\beta}{3}|\psi^+\rangle|11\rangle$ or $\frac{\sqrt{2}\alpha}{3}|\psi^+\rangle|00\rangle + \frac{\sqrt{2}\beta}{3}|11\rangle|\psi^+\rangle$, then the two 3-qubit entangled state described by the density operators ρ_{146} and ρ_{325} are different and the broadcasting is possible for $0.6 < \alpha^2 < 1$ or $0.14 <$

$\alpha^2 < 0.4$ according to the measurement results. Also if the outcome of the measurement is $\frac{\alpha}{3}|\psi^+\rangle|\psi^+\rangle + \frac{2\beta}{3}|1111\rangle$, then the state described by the density operators ρ_{146} and ρ_{325} are identical and the broadcasting is possible for $0.38 < \alpha^2 < 0.73$.

3.6 Secret generation of two 3-qubit entangled state between three distant partners

In this section, we investigate a question, can we secretly generate two 3-qubit entangled state shared between three distant partners using LOCC? The answer is in affirmative. To generate three-qubit entangled states between three distant partners, we require only two well-known concept: (i) quantum cloning and (ii) entanglement swapping [17, 158].

Let us suppose for the implementation of any particular cryptographic scheme, three distant partners Alice, Bob and Carol want to generate two three qubit entangled states between them. To do the same task, let us assume that initially Alice-Bob and Carol-Alice share two qubit entangled states described by the density operators ρ_{13} , ρ_{78} , where Alice has qubits '1' and '8', Bob and Carol possess qubits '3' and '7' respectively. Then Alice and Bob adopting the broadcasting process described in the previous section to generate two three-qubit entangled state in between them. Therefore, Alice and Bob now have two 3-qubit entangled states described by the density operators ρ_{146} and ρ_{325} where Alice has qubits '1', '2' and '5' and Bob possesses '3', '4' and '6'. Now we are in a position for the illustration of the generation of 3-qubit entangled state between three parties at distant places by using the concept of entanglement swapping.

Without any loss of generality, we take a three-qubit entangled state between two distant parties described by the density operator ρ_{325} .

The density operator ρ_{325} can be rewritten as

$$\begin{aligned} \rho_{325} = & \frac{1}{N} \left[\frac{4\alpha^2}{9} \left(\frac{2}{3} |000\rangle\langle 000| + \frac{1}{3} |0\psi^+\rangle\langle 0\psi^+| \right) + \frac{\alpha\beta^*}{9} \left(\frac{\sqrt{2}}{3} |000\rangle\langle 1\psi^+| + \frac{\sqrt{2}}{3} |0\psi^+\rangle\langle 111| \right) + \right. \\ & \frac{\alpha\beta}{9} \left(\frac{\sqrt{2}}{3} |111\rangle\langle 0\psi^+| + \frac{\sqrt{2}}{3} |1\psi^+\rangle\langle 000| \right) + \frac{|\beta|^2}{36} \left(\frac{2}{3} |011\rangle\langle 011| + \frac{2}{3} |0\psi^+\rangle\langle 0\psi^+| + \right. \\ & \left. \left. \frac{2}{3} |000\rangle\langle 000| + \frac{2}{3} |111\rangle\langle 111| + \frac{2}{3} |1\psi^+\rangle\langle 1\psi^+| + \frac{2}{3} |100\rangle\langle 100| \right) \right] \end{aligned} \quad (3.71)$$

where qubits 2 and 5 are possessed by Alice and qubit 3 is possessed by Bob respectively. To achieve the goal of the generation of three qubit entangled state between three distant partners, we proceed in the following way:

Let Alice and Carol shared a singlet state

$$|\psi^-\rangle_{87} = \left(\frac{1}{\sqrt{2}}\right)(|01\rangle - |10\rangle) \quad (3.72)$$

where particles 8 and 7 are possessed by Alice and Carol respectively.

The combined state between Alice, Bob and Carol is given by the

$$\rho_{32587} = \rho_{325} \otimes |\psi^-\rangle_{78}\langle\psi^-| \quad (3.73)$$

Alice then performs Bell state measurement on the particles 2 and 8 in the basis $\{|B_1^\pm\rangle, |B_2^\pm\rangle\}$, where $|B_1^\pm\rangle = \left(\frac{1}{\sqrt{2}}\right)(|00\rangle \pm |11\rangle)$, $|B_2^\pm\rangle = \left(\frac{1}{\sqrt{2}}\right)(|01\rangle \pm |10\rangle)$

If the measurement result is $|B_1^+\rangle$, then the 3-qubit density operator is given by

$$\begin{aligned} \rho_{357} = & \frac{1}{N} \left[\frac{4\alpha^2}{9} \left[\frac{2}{3} |001\rangle\langle 001| + \frac{1}{6} (|011\rangle\langle 011| - |011\rangle\langle 000| - |000\rangle\langle 011| + |000\rangle\langle 000|) \right] + \right. \\ & \frac{\alpha\beta^*}{27} (|001\rangle\langle 111| - |001\rangle\langle 100| + |000\rangle\langle 110| - |011\rangle\langle 110|) + \frac{\alpha\beta}{27} (-|110\rangle\langle 011| + \\ & |110\rangle\langle 000| + |111\rangle\langle 001| - |100\rangle\langle 001|) + \frac{|\beta|^2}{36} \left[\frac{2}{3} (|010\rangle\langle 010| + |001\rangle\langle 001| + \right. \\ & |110\rangle\langle 110| + |101\rangle\langle 101|) + \frac{1}{3} (|011\rangle\langle 011| - |011\rangle\langle 000| - |000\rangle\langle 011| + |000\rangle\langle 000| \\ & \left. \left. + |111\rangle\langle 111| - |111\rangle\langle 100| - |100\rangle\langle 111| + |100\rangle\langle 100|) \right] \right] \quad (3.74) \end{aligned}$$

After Bell-state measurement, Alice announces publicly the measurement result. Thereafter, Alice, Bob and Carol operate an unitary operator $U_1 = I_3 \otimes (\sigma_z)_5 \otimes (\sigma_x)_7$ on their respective particles to retrieve the state described by the density operator ρ_{325} .

If the measurement result is $|B_1^-\rangle$ or $|B_2^+\rangle$ or $|B_2^-\rangle$ then accordingly they operate an unitary operator $U_2 = I_3 \otimes (I_5) \otimes (\sigma_x)_7$ or $U_3 = I_3 \otimes (I_5) \otimes (\sigma_z)_7$ or $U_4 = I_3 \otimes (I_3) \otimes (I_7)$ on their respective particles to retrieve the state described by the density operator ρ_{325} . For the remaining cases given in step-3 of the previous section 3.5, we can proceed in a similar manner as above. Hence, in every cases we find that after getting the measurement result, each party (Alice, Bob and Carol) apply suitable unitary operators on their respective particles to share the 3-qubit entangled state in between them, which is

previously shared between only two distant partners Alice and Bob. The above protocol is described pictorially in figures 3.5 and 3.6.

Therefore, we describe here the secret generation of 3-qubit entangled state between three distant partners starting from 3-qubit entangled state shared between two distant partners using quantum cloning and entanglement swapping. This quantum channel generated by the above procedures can be regarded as a secret quantum channel because the result of the measurement on the machine state vectors are transmitted secretly by quantum cryptographic scheme.

Chapter 4

On universal quantum deletion transformation

The most remarkable discovery ever made by scientists was science itself - Jacob Bronowski

The whole of science is nothing more than a refinement of everyday thinking - Albert Einstein

Science cannot solve the ultimate mystery of Nature. And it is because in the last analysis we ourselves are part of the mystery we are trying to solve - Max Planck

4.1 *Prelude*

The complementary theory of "quantum no-cloning theorem" is the "quantum no-deleting" principle [118]. It states that linearity of quantum theory forbids deleting one unknown quantum state against a copy in either a reversible or an irreversible manner. We can understand the principle behind quantum deletion more clearly, if we compare quantum deletion with the "Landauer erasure principle" [103]. It tells us that a single copy of some classical information can be erased with some energy cost. It is an irreversible operation. In quantum theory, erasure of a single unknown state may be thought of as swapping it with some standard state and then dumping it into the environment. Unlike quantum erasure, quantum deletion is a different concept. Quantum deletion [90, 119] is more like reversible 'uncopying' of an unknown quantum state. Al-

though there is not a perfect deleting machine, the corresponding no-deleting principle does not prohibit us from constructing the approximate deleting machine [1, 2, 4, 119]. If quantum deleting could be done, then one would create a standard blank state onto which an unknown quantum state is copied approximately by deterministic cloning or exactly by probabilistic cloning process [50, 116, 155]. We can apply the quantum deleting machine in a situation when scarcity of memory in a quantum computer occurs.

In this chapter, we discuss the problem of constructing an efficient universal deletion machine in the sense of high fidelity of deletion. We construct two types of "universal quantum deletion machine" which approximately deletes a copy such that the fidelity of deletion does not depend on the input state. Also we classify two types of universal quantum deletion machines: (1) a conventional deletion machine described by one unitary operator viz. Deleter and (2) a modified deletion machine described by two unitary operators viz. Deleter and Transformer. Here it is shown that the modified deletion machine deletes a qubit with fidelity $\frac{3}{4}$, which is the maximum limit for deleting an unknown quantum state. In addition to this we also show that the modified deletion machine retains the qubit in the first mode with average fidelity 0.77 (approx.) which is slightly greater than the fidelity of measurement for two given identical states [108]. We also show that the deletion machine itself is input state independent, i.e., the information is not hidden in the deleting machine, and hence we can delete the information completely from the deletion machine.

Next, we study the quantum deletion machine with two transformers, and show that the deletion machine with a single transformer performs better than the deletion machine with more than two transformers. We also observe that the fidelity of deletion depends on the blank state used in the deleter, and so for different blank states the fidelity is different. Furthermore, we study the Pati-Braunsein deleter with transformer.

This chapter is based on our works "Quantum deletion: Beyond no-deletion principle" [2] and "Improving the fidelity of deletion" [4]

4.2 Quantum deletion machines

In this section, we discuss two types of deletion machines. The first type of deletion machine is conventional i.e. it just deletes a qubit while the second type of deletion machine not only deletes a qubit but also transforms the state after deletion operation. The newly defined deletion machine i.e. the modified deletion machine, consists of two parts, the deleter and the transformer.

1. Deleter: This is nothing but a unitary transformation U used to delete one copy from among two given copies of an unknown quantum state.

A unitary transformation U which describes a deleter is given below:

$$U|00\rangle_{ab}|A\rangle_c \rightarrow |0\rangle_a|\Sigma\rangle_b|A_0\rangle_c + [|0\rangle_a|1\rangle_b + |1\rangle_a|0\rangle_b]|B_0\rangle_c \quad (4.1)$$

$$U|01\rangle_{ab}|A\rangle_c \rightarrow |0\rangle_a|\Sigma_\perp\rangle_b|D_0\rangle_c + |1\rangle_a|0\rangle_b|C_0\rangle_c \quad (4.2)$$

$$U|10\rangle_{ab}|A\rangle_c \rightarrow |1\rangle_a|\Sigma\rangle_b|D_0\rangle_c + |0\rangle_a|1\rangle_b|C_0\rangle_c \quad (4.3)$$

$$U|11\rangle_{ab}|A\rangle_c \rightarrow |1\rangle_a|\Sigma_\perp\rangle_b|A_1\rangle_c + [|0\rangle_a|1\rangle_b + |1\rangle_a|0\rangle_b]|B_1\rangle_c \quad (4.4)$$

where $|A\rangle$ is the initial and $|A_i\rangle, |B_i\rangle, |C_j\rangle, |D_j\rangle$ ($i=0,1; j=0$) are the final machine state vectors. $|\Sigma\rangle$ is some standard state and $|\Sigma_\perp\rangle$ denotes a state orthogonal to $|\Sigma\rangle$.

We assume

$$\langle A_0|B_0\rangle = \langle A_0|D_0\rangle = \langle A_1|D_0\rangle = \langle A_1|B_1\rangle = \langle A_0|A_1\rangle = \langle B_0|C_0\rangle = \langle B_0|D_0\rangle = 0, \quad (4.5)$$

$$\langle A_0|B_1\rangle = \langle A_1|B_0\rangle = \langle B_0|B_1\rangle = \langle B_1|D_0\rangle = \langle C_0|A_1\rangle = \langle B_1|C_0\rangle = 0,$$

$$\langle A|A_0\rangle = \langle A|D_0\rangle = \langle A|A_1\rangle = Y, \langle A|B_0\rangle = \langle A|C_0\rangle = \langle A|B_1\rangle = 0. \quad (4.6)$$

The normalization condition of the transformation (4.1-4.4) gives

$$\begin{aligned} \langle A_i|A_i\rangle + 2\langle B_i|B_i\rangle &= 1, \quad i = 0, 1 \\ \langle C_0|C_0\rangle + \langle D_0|D_0\rangle &= 1 \end{aligned} \quad (4.7)$$

The orthogonality condition to be satisfied for the transformation (4.1-4.4) is

$$\langle A_0|C_0\rangle = \langle D_0|C_0\rangle = 0 \quad (4.8)$$

2. Transformer: It is described by a unitary transformation T . It is used in the deletion machine to increase the fidelity of deletion and minimize the distortion of the undeleted qubit.

The unitary operator T [84] is defined by

$$T = |\psi^+\rangle\langle 00| + |11\rangle\langle 01| + |\psi^-\rangle\langle 10| + |00\rangle\langle 11| \quad (4.9)$$

where $|\psi^\pm\rangle = (\frac{1}{\sqrt{2}})(|01\rangle \pm |10\rangle)$

A few Definitions:

Let $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ with $\alpha^2 + |\beta|^2 = 1$ be any unknown quantum state.

Without any loss of generality we can take α real and β complex.

Let ρ_a and ρ_b be the reduced density operators describing the state of the undeleted qubit in mode 'a' and the state of the deleted qubit in mode 'b' respectively and ρ_c denote the density operator of the machine state after the deletion operation.

Let $F_a = \langle\psi|\rho_a|\psi\rangle$, $F_b = \langle\Sigma'|\rho_b|\Sigma'\rangle$, where $|\Sigma'\rangle = (\frac{1}{\sqrt{2}})(|\Sigma\rangle + |\Sigma_\perp\rangle)$ denotes the fidelity of the qubit in the modes a and b, respectively, after deletion operation and $F_c = \langle A|\rho_c|A\rangle$ denotes the overlapping between the initial and final machine state vectors.

Definition 4.1: (State dependent deletion machine) A deletion machine is said to be state dependent if F_a , F_b and F_c depend on the input state.

Definition 4.2: (Universal deletion machine) A deletion machine is said to be universal if F_b and F_c are independent of the input state. The machine is optimal if it maximizes F_a and F_b .

Definition 4.3: (Ideal deletion machine) A deletion machine is said to be ideal if F_a , F_b and F_c are input state independent and the machine is optimal if it maximizes F_a and F_b .

Note: From definition 4.2 and 4.3, we can say that every ideal deletion machine is universal but converse is not true.

4.3 Conventional deletion machine (deletion machine without transformer)

In this section, we show that the conventional deletion machine (4.1-4.4), i.e., a deletion machine without transformer, becomes either a universal deletion machine or an ideal deletion machine in some restricted cases. The conventional deletion machine is described in figure-4.1.

The deletion machine can be shown to be universal or ideal following three steps:

Step 1: The reduced density operator in the mode '1' is given by

$$\begin{aligned} \rho_1 = \text{Tr}_{23}(\rho_{123}) &= |0\rangle\langle 0|[\alpha^4(\langle A_0|A_0\rangle + \langle B_0|B_0\rangle) + \alpha^2|\beta|^2 + |\beta|^4\langle B_1|B_1\rangle] + \\ &|1\rangle\langle 1|[\alpha^4\langle B_0|B_0\rangle + \alpha^2|\beta|^2 + |\beta|^4(\langle A_1|A_1\rangle + \langle B_1|B_1\rangle)] \end{aligned} \quad (4.10)$$

Let us assume

$$\langle A_0|A_0\rangle = \langle A_1|A_1\rangle = \langle D_0|D_0\rangle = 1 - 2\lambda \quad (4.11)$$

$$\langle B_0|B_0\rangle = \langle B_1|B_1\rangle = \frac{\langle C_0|C_0\rangle}{2} = \lambda \quad (4.12)$$

with $0 \leq \lambda \leq \frac{1}{2}$ which follows from the Schwarz inequality.

$$\rho_1 = |0\rangle\langle 0|[\alpha^4(1 - \lambda) + \alpha^2|\beta|^2 + |\beta|^4\lambda] + |1\rangle\langle 1|[\alpha^4\lambda + \alpha^2|\beta|^2 + |\beta|^4(1 - \lambda)] \quad (4.13)$$

The overlapping between the input state $|\psi\rangle$ and the density operator ρ_1 is given by

$$F_1 = \langle \psi | \rho_1 | \psi \rangle = (1 - \lambda) + 2\alpha^2(1 - \alpha^2)(2\lambda - 1) \quad (4.14)$$

Therefore F_1 depends on α^2 and the parameter λ .

Now it is interesting to discuss results for two different values of λ in two different cases. In the first case F_1 is found to be input state independent and in the second case it depends on the input state.

Case I: If $\lambda \rightarrow \frac{1}{2}$ then $F_1 \rightarrow \frac{1}{2}$. Although we are able to make F_1 input state independent, the performance of the deletion machine is not very satisfactory since it fails to retain the qubit in the first mode faithfully after the deletion operation.

Case II: If $\lambda \rightarrow 0$, then $F_1 \rightarrow 1 - 2\alpha^2(1 - \alpha^2)$, which is input state dependent and

therefore we have to calculate the average value.

The average fidelity is given by

$$\overline{F_1} = \int_0^1 F_1(\alpha^2) d\alpha^2 \rightarrow \frac{2}{3} \quad (4.15)$$

This value is equal to the fidelity of measurement for a given single unknown state. Although F_1 is input state dependent the average value $\overline{F_1}$ exceeds the fidelity discussed in case I.

Step 2: The reduced density operator in the mode 2 is given by

$$\begin{aligned} \rho_2 = & Tr_{13}(\rho_{123}) = |0\rangle\langle 0|[\alpha^4\langle B_0|B_0\rangle + \alpha^2|\beta|^2\langle C_0|C_0\rangle + |\beta|^4\langle B_1|B_1\rangle] + \\ & |1\rangle\langle 1|[\alpha^4\langle B_0|B_0\rangle + \alpha^2|\beta|^2\langle C_0|C_0\rangle + |\beta|^4\langle B_1|B_1\rangle] + |\Sigma\rangle\langle\Sigma|[\alpha^4\langle A_0|A_0\rangle + \\ & \alpha^2|\beta|^2\langle D_0|D_0\rangle] + |\Sigma_\perp\rangle\langle\Sigma_\perp|[\beta^4\langle A_1|A_1\rangle + \alpha^2|\beta|^2\langle D_0|D_0\rangle] \end{aligned} \quad (4.16)$$

Using equations (4.11) and (4.12), equation (4.16) reduces to

$$\begin{aligned} \rho_2 = & |0\rangle\langle 0|[\alpha^4\lambda + 2\alpha^2|\beta|^2\lambda + |\beta|^4\lambda] + |1\rangle\langle 1|[\alpha^4\lambda + 2\alpha^2|\beta|^2\lambda + |\beta|^4\lambda] + \\ & |\Sigma\rangle\langle\Sigma|[\alpha^2(1-2\lambda)] + |\Sigma_\perp\rangle\langle\Sigma_\perp|[\beta^2(1-2\lambda)] \end{aligned} \quad (4.17)$$

The fidelity of deletion is defined by

$$F_2 = \langle\Sigma'|\rho_2|\Sigma'\rangle = \left(\frac{1}{2}\right)[(1-2\lambda) + (K_1 + K_2)\lambda] \quad (4.18)$$

where

$$K_1 = \langle\Sigma|0\rangle^2 + |\langle\Sigma|1\rangle|^2 + \langle\Sigma|0\rangle\langle 0|\Sigma_\perp\rangle + \langle\Sigma|1\rangle\langle 1|\Sigma_\perp\rangle \quad (4.19)$$

$$K_2 = |\langle\Sigma_\perp|0\rangle|^2 + \langle\Sigma_\perp|1\rangle^2 + \langle\Sigma|0\rangle\langle 0|\Sigma_\perp\rangle + \langle\Sigma|1\rangle\langle 1|\Sigma_\perp\rangle \quad (4.20)$$

The standard state $|\Sigma\rangle$ can be written as $|\Sigma\rangle = m_1|0\rangle + m_2|1\rangle$, where without any loss of generality we can take m_1 real and m_2 complex satisfying the relation

$$m_1^2 + |m_2|^2 = 1. \quad (4.21)$$

A state orthogonal to $|\Sigma\rangle$ is given by

$$|\Sigma_\perp\rangle = -m_2^*|0\rangle + m_1|1\rangle \quad (4.22)$$

Therefore,

$$\langle \Sigma|0\rangle = \langle \Sigma_{\perp}|1\rangle = m_1, \langle \Sigma|1\rangle = m_2^*, \langle \Sigma_{\perp}|0\rangle = -m_2 \quad (4.23)$$

Using equations (4.19), (4.20), (4.21) and (4.23), we get

$$K_1 + K_2 = 2 \quad (4.24)$$

Putting the value of $(K_1 + K_2)$ in equation (4.18), we get

$$F_2 = \frac{1}{2} \quad (4.25)$$

Here we note that the fidelity of deletion neither depends on input state nor on machine state. The value of the fidelity of deletion is calculated to be $\frac{1}{2}$, which is not a very satisfactory result at all. The same value of the fidelity is also obtained by Qiu [126] for his deletion machine and it emphasizes the difficulty of improving its fidelity. We also find here that the fidelity of deletion for our prescribed deletion machine cannot be improved further if the machine is kept in its present form but the fidelity may be improved if we define a deletion machine in another way, which we discuss in details in the next section.

Step 3: The reduced density operator in the mode '3' is given by

$$\begin{aligned} \rho_3 = Tr_{12}(\rho_{123}) = & \alpha^4(|A_0\rangle\langle A_0| + m_2^*|A_0\rangle\langle B_0| + m_2|B_0\rangle\langle A_0| + 2|B_0\rangle\langle B_0|) + \\ & \alpha^3\beta^*(m_2|A_0\rangle\langle C_0| + 2m_1|B_0\rangle\langle D_0| + 2|B_0\rangle\langle C_0|) + \alpha^3\beta(m_2^*|C_0\rangle\langle A_0| \\ & + 2m_1|D_0\rangle\langle B_0| + 2|C_0\rangle\langle B_0|) + \alpha^2|\beta|^2[m_2|A_0\rangle\langle B_1| + m_2^*|B_1\rangle\langle A_0| - \\ & m_2^*|A_1\rangle\langle B_0| - m_2|B_0\rangle\langle A_1| + 2(|B_0\rangle\langle B_1| + |B_1\rangle\langle B_0|) + 2(|C_0\rangle\langle C_0| + \\ & |D_0\rangle\langle D_0|) + 2m_1(|C_0\rangle\langle D_0| + |D_0\rangle\langle C_0|)] + \alpha|\beta|^2\beta[2m_1(|B_1\rangle\langle D_0| + \\ & |D_0\rangle\langle B_1|) - m_2^*|A_1\rangle\langle C_0| - m_2|C_0\rangle\langle A_1| + 2(|B_1\rangle\langle C_0| + |C_0\rangle\langle B_1|)] + \\ & |\beta|^4(|A_1\rangle\langle A_1| - m_2^*|A_1\rangle\langle B_1| - m_2|B_1\rangle\langle A_1| + 2|B_1\rangle\langle B_1|) \end{aligned} \quad (4.26)$$

Using equation (4.6), (4.26) and the relation $\alpha^2 + |\beta|^2 = 1$, we get

$$\langle A|\rho_3|A\rangle = Y^2 \quad (4.27)$$

which is independent of α^2 . This means that the information is not hidden in the deletion machine and hence it deletes the state completely because we cannot retrieve the state by applying a unitary transformation from the deletion machine.

Note: (1) If $\lambda \rightarrow \frac{1}{2}$, then F_1 , F_2 and $\langle A|\rho_3|A\rangle$ are independent of α^2 . Also $F_1 \rightarrow \frac{1}{2}$, $F_2 = \frac{1}{2}$. Therefore, for $\lambda \rightarrow \frac{1}{2}$, the conventional deletion machine becomes ideal deletion machine in the limiting sense but the machine is not optimal.

(2) If $\lambda \neq \frac{1}{2}$ then also F_2 and $\langle A|\rho_3|A\rangle$ are independent of α^2 because they do not depend on λ so the conventional deletion machine becomes universal deletion machine for all values of λ ($0 \leq \lambda < \frac{1}{2}$).

Now case-2 is interesting in the sense that if $\lambda \rightarrow 0$, then the average value of F_1 tends to the maximum limit $\frac{2}{3}$ that is also obtained by state dependent Pati-Braunstein deletion machine. Moreover, the fidelity of a qubit in mode 1, i.e., F_1 is found to be greater than the fidelity of deletion F_2 .

4.4 *Modified deletion machine (deletion machine with single transformer)*

In the preceding section 4.3, we discussed the deletion machine without considering a vital part of it. In this section we take into account that important part of the deletion machine without which we cannot improve the fidelity of deletion. In addition to a unitary transformation U (named the deleter) that deletes a qubit, a unitary operator T (named the transformer) must be used in the deletion machine. The role of the transformer is to transform the resultant state immediately obtained after the deletion operation, thereby improving the fidelity of deletion of the qubit in the second mode and increasing the fidelity of the retained qubit in the first mode. The modified deletion machine is described in figure-4.2.

In the first chamber the deletion process is completed. Thereafter the deleted state described by the density operator ρ_{123} enters into the second chamber where another

unitary operator called transformer transforms it into the state ρ'_{123} ,

$$\rho'_{123} = (I \otimes T)\rho_{123}(I \otimes T)^\dagger \quad (4.28)$$

The reduced density operator describing the state ρ'_1 is given by

$$\begin{aligned} \rho'_1 = \text{Tr}_{23}(\rho'_{123}) = & |0\rangle\langle 0|(\frac{1}{2})(\alpha^4[m_1^2(1-2\lambda) + \lambda] + \alpha^2|\beta|^2\{[3|m_2|^2 - m_1(m_2 + m_2^*) \\ & + m_1^2](1-2\lambda) + 2\lambda\} + |\beta|^4[(|m_2|^2 + 2m_1^2)(1-2\lambda) + \lambda]) + |0\rangle\langle 1|(\frac{1}{\sqrt{2}}) \times \\ & (\alpha^4[m_1m_2^*(1-2\lambda) + \lambda] + \alpha^2|\beta|^2\{2\lambda + [m_1^2 - m_2^2 - m_1(m_2 + m_2^*)](1-2\lambda)\} + \\ & |\beta|^4[\lambda + m_1m_2(1-2\lambda)]) + |1\rangle\langle 0|(\frac{1}{\sqrt{2}})(\alpha^4[m_1m_2(1-2\lambda) + \lambda] + \alpha^2|\beta|^2\{2\lambda + \\ & [m_1^2 - (m_2^*)^2 - m_1(m_2 + m_2^*)](1-2\lambda)\} + |\beta|^4[\lambda + m_1m_2^*(1-2\lambda)]) + \\ & |1\rangle\langle 1|(\frac{1}{2})(\alpha^4[(m_1^2 + 2|m_2|^2)(1-2\lambda) + 3\lambda] + \alpha^2|\beta|^2\{[|m_2|^2 + m_1(m_2 + m_2^*) + \\ & 3m_1^2](1-2\lambda) + 6\lambda\} + |\beta|^4[|m_2|^2(1-2\lambda) + 3\lambda]). \end{aligned} \quad (4.29)$$

The fidelity of the qubit in mode 1 is given by

$$F_3 = \langle \psi | \rho'_1 | \psi \rangle \rightarrow \frac{3}{4} - \frac{\alpha^2}{2} + \frac{\alpha(\beta + \beta^*)}{2\sqrt{2}} \quad \text{for } \lambda \rightarrow \frac{1}{2} \quad (4.30)$$

If β is real, then the average fidelity of this mode is

$$\overline{F_3} = \int_0^1 F_3(\alpha^2) d\alpha^2 \rightarrow \frac{1}{2} + \frac{\pi}{8\sqrt{2}} = 0.77 \quad (\text{approx.}) \quad (4.31)$$

The state described by the reduced density operator ρ'_2 is given by

$$\begin{aligned} \rho'_2 = \text{Tr}_{13}(\rho'_{123}) = & |0\rangle\langle 0|(\frac{1}{2})(\alpha^4[m_1^2(1-2\lambda) + \lambda] + \alpha^2|\beta|^2\{[3|m_2|^2 + m_1(m_2 + m_2^*) \\ & + m_1^2](1-2\lambda) + 2\lambda\} + |\beta|^4[(|m_2|^2 + 2m_1^2)(1-2\lambda) + \lambda]) + |0\rangle\langle 1|(\frac{1}{\sqrt{2}}) \times \\ & (\alpha^4[m_1m_2^*(1-2\lambda) - \lambda] - \alpha^2|\beta|^2\{2\lambda + [m_1^2 + m_2^2 + m_1(m_2^* - m_2)](1-2\lambda)\} \\ & - |\beta|^4[\lambda + m_1m_2(1-2\lambda)]) + |1\rangle\langle 0|(\frac{1}{\sqrt{2}})(\alpha^4[m_1m_2(1-2\lambda) - \lambda] - \alpha^2|\beta|^2\{2\lambda + \\ & [m_1^2 + (m_2^*)^2 + m_1(m_2 - m_2^*)](1-2\lambda)\} - |\beta|^4[\lambda + m_1m_2^*(1-2\lambda)]) + \\ & |1\rangle\langle 1|(\frac{1}{2})(\alpha^4[(m_1^2 + 2|m_2|^2)(1-2\lambda) + 3\lambda] + \alpha^2|\beta|^2\{[|m_2|^2 - m_1(m_2 + m_2^*) + \\ & 3m_1^2](1-2\lambda) + 6\lambda\} + |\beta|^4[|m_2|^2(1-2\lambda) + 3\lambda]). \end{aligned} \quad (4.32)$$

The fidelity of the qubit in mode 2 is given by

$$\begin{aligned} F_4 = \langle \Sigma' | \rho'_2 | \Sigma' \rangle = & \frac{1}{2}[R_1(m_1 - m_2)(m_1 - m_2^*) + R_2(m_1 + m_2)(m_1 + m_2^*) + \\ & R_3(m_1 - m_2)(m_1 + m_2) + R_4(m_1 - m_2^*)(m_1 + m_2^*)], \end{aligned} \quad (4.33)$$

where

$$R_1 = \left(\frac{1}{2}\right)\{\alpha^4[m_1^2(1-2\lambda) + \lambda] + \alpha^2|\beta|^2\{[3|m_2|^2 - m_1(m_2 + m_2^*) + m_1^2](1-2\lambda) + 2\lambda\} + |\beta|^4[(|m_2|^2 + 2m_1^2)(1-2\lambda) + \lambda]\} \quad (4.34)$$

$$R_2 = \left(\frac{1}{2}\right)\{\alpha^4[m_1^2 + 2|m_2|^2(1-2\lambda) + 3\lambda] + \alpha^2|\beta|^2\{[|m_2|^2 + m_1(m_2 + m_2^*) + 3m_1^2](1-2\lambda) + 6\lambda\} + |\beta|^4[|m_2|^2(1-2\lambda) + 3\lambda]\} \quad (4.35)$$

$$R_3 = \left(\frac{1}{\sqrt{2}}\right)\{\alpha^4[m_1m_2^*(1-2\lambda) - \lambda] - \alpha^2|\beta|^2\{[m_1^2 + m_2^2 + m_1(m_2^* - m_2)] \times (1-2\lambda) + 2\lambda\} - |\beta|^4[m_1m_2(1-2\lambda) + \lambda]\} \quad (4.36)$$

$$R_4 = \left(\frac{1}{\sqrt{2}}\right)\{\alpha^4[m_1m_2(1-2\lambda) - \lambda] - \alpha^2|\beta|^2\{[m_1^2 + (m_2^*)^2 + m_1(m_2 - m_2^*)] \times (1-2\lambda) + 2\lambda\} - |\beta|^4[m_1m_2^*(1-2\lambda) + \lambda]\} \quad (4.37)$$

If $m_1 = m_2 = \frac{1}{\sqrt{2}}$, then the expression for F_4 given in equation (4.33) reduces to

$$F_4 = R_2 \rightarrow \frac{3}{4} = 0.75 \quad \text{for } \lambda \rightarrow \frac{1}{2} \quad (4.38)$$

Since the machine states are invariant under the unitary transformation T , so $\langle A|\rho'_3|A\rangle = \langle A|\rho_3|A\rangle = Y^2$, which is independent of α^2 .

Hence the deletion machine with transformer becomes a universal deletion machine when the machine parameter $\lambda \rightarrow \frac{1}{2}$ and $m_1 = m_2 = \frac{1}{\sqrt{2}}$. This universal deletion machine deletes a qubit with fidelity $\frac{3}{4}$ (in the limiting sense), which is the maximum limit for deleting an unknown qubit. In addition, the average fidelity of the qubit in the first mode is found to be 0.77, which is greater than the average fidelity ($\overline{F}_a = 0.66$) obtained by Pati-Braunstein deletion machine.

Furthermore, if the machine parameter λ tends to $\frac{1}{2}$ then for all real blank state parameters m_1 and m_2 the limiting fidelity of deletion F_4 given in equation (4.33) goes towards F'_4 i.e.

$$F_4 \rightarrow F'_4 = \frac{1}{2}\left[1 + m_1m_2 - \frac{m_1^2 - m_2^2}{\sqrt{2}}\right], \quad \text{as } \lambda \rightarrow \frac{1}{2} \quad (\text{for real } m_1 \text{ and } m_2) \quad (4.39)$$

Equation (4.39) shows that the fidelity of deletion remains the same for all input states α^2 and it depends only on the blank state. Since the limiting fidelity of deletion of the deletion machine with one transformer depends on the parameters m_1 and m_2 so the

variation of the limiting fidelity of deletion with m_1 and m_2 is studied and given in the table 4.1.

Table-4.1: Limiting fidelities for deletion machine with one transformer

m_1^2	m_2^2	$m_1^2 - m_2^2$	Limiting fidelity(F'_4) = $\frac{1}{2}[1 \pm \frac{\sqrt{1-(m_1^2-m_2^2)}}{2} - \frac{m_1^2-m_2^2}{\sqrt{2}}]$ according as $m_1 m_2 > 0$ or $m_1 m_2 < 0$ (up to two significant figures)
0.0	1.0	-1.0	0.85
0.1	0.9	-0.8	0.93 or 0.63
0.2	0.8	-0.6	0.91 or 0.51
0.3	0.7	-0.4	0.87 or 0.41
0.4	0.6	-0.2	0.81 or 0.32
0.5	0.5	0.0	0.75
0.6	0.4	0.2	0.67 or 0.18
0.7	0.3	0.4	0.58 or 0.12
0.8	0.2	0.6	0.48 or 0.08
0.9	0.1	0.8	0.36 or 0.06
1.0	0.0	1.0	0.14

Illustration of the table 4.1:

If the blank state is of the form $|\Sigma\rangle = \sqrt{0.1}|0\rangle + \sqrt{0.9}|1\rangle$, then the deletion machine with one transformer deletes a qubit with fidelity 0.93 and if the blank state either take the form $|\Sigma\rangle = -\sqrt{0.1}|0\rangle + \sqrt{0.9}|1\rangle$ or $|\Sigma\rangle = \sqrt{0.1}|0\rangle - \sqrt{0.9}|1\rangle$, then the fidelity of deletion is found to be 0.63.

Therefore, we can observe from the above table that if the product of the parameter of the blank state is negative (i.e. when either m_1 or m_2 is negative), then the deletion machine deletes the state with lower fidelity of deletion but if the product of the parameter of the blank state is positive (i.e. when both m_1 and m_2 are negative or positive), then the deletion machine performs well in the sense of high fidelity of deletion. In this work we have discussed the quantum deletion machine with one transformer for various values of the parameters m_1 and m_2 and find that the quantum deletion machine really works well for some blank states and, with the help of those blank states, quantum deletion machine deletes a quantum state with fidelity of deletion higher than 0.75.

4.5 Quantum deletion machine with two transformers

In this section, we study the quantum deletion machine (4.1-4.4) with two transformers. Since the introduction of the transformer increases the fidelity of deletion so one may expect that the application of a transformer more than one time increases the fidelity of deletion further and therefore there may exist a threshold number of the transformers (i.e. maximum number of transformers) whose application on the deleted state increases the fidelity of deletion to its optimal value. But, we will show that this is not necessarily true.

Let $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ with $\alpha^2 + |\beta|^2 = 1$ be any unknown quantum state, where α is real and β is complex.

The modified deletion machine with two transformers deletes one of the copies of an input state $|\psi\rangle|\psi\rangle$ and then, after transforming the deleted qubit, the final output state of the deletion machine is described by the density operator $\rho_{12c}^{out} = (T)^2|\chi_d^{out}\rangle_{12c}\langle\chi_d^{out}|(T^\dagger)^2$ where $|\chi_d^{out}\rangle$ represents a state after passing through the deleter (4.1-4.4). Since we are interested to see the performance of the deletion machine with two transformers in the sense of how well it deletes a qubit, we only consider the state of the qubit in mode 2. Therefore, the reduced density operator describes the state of the qubit in mode 2 is given by ρ_2^{out} . But as, $\lambda \rightarrow \frac{1}{2}$, $\rho_2^{out} \rightarrow \rho_2'^{out}$ where

$$\rho_2'^{out} = \frac{5}{8}|0\rangle\langle 0| + \frac{1}{4}\left(\frac{1}{\sqrt{2}} - 1\right)|0\rangle\langle 1| + \frac{1}{4}\left(\frac{1}{\sqrt{2}} - 1\right)|1\rangle\langle 0| + \frac{3}{8}|1\rangle\langle 1| \quad (4.40)$$

Equation (4.40) shows that the state described by the density operator $\rho_2'^{out}$ is input state independent i.e. whatever be the input quantum state, after passing through the deleter and two transformers, the resulting output state remains the same.

The limiting fidelity of deletion is given by

$$F = \langle \Sigma' | \rho_2'^{out} | \Sigma' \rangle, \text{ where } |\Sigma'\rangle = \left(\frac{1}{\sqrt{2}}\right)(|\Sigma\rangle + |\Sigma_\perp\rangle) \quad (4.41)$$

Hence,

$$F = \frac{1}{2} \left[\frac{5}{8} (1 + m_1 m_2^* - m_1 m_2) + \frac{1}{4} \left(\frac{1}{\sqrt{2}} - 1 \right) (2m_1^2 - (m_2^*)^2 - m_2^2) + \frac{3}{8} (1 + m_1 m_2^* + m_1 m_2) \right] \quad (4.42)$$

Equation (4.42) shows that the fidelity of deletion varies as m_1 and m_2 , i.e., the limiting fidelity depends on the blank state used in the deleter but not on the arbitrary input state.

In particular, if we assume m_1 and m_2 to be real, then the variation of fidelity with the amplitudes of the blank state m_1 and m_2 is given in the table-4.2:

Table-4.2: Limiting fidelities for deletion machine with two transformers

m_1^2	m_2^2	$m_1^2 - m_2^2$	Limiting fidelity(F) = $\frac{1}{2} [1 \mp \frac{\sqrt{1-(m_1^2-m_2^2)^2}}{4} + (\frac{1}{2})(\frac{1}{\sqrt{2}} - 1)(m_1^2 - m_2^2)]$ according as $m_1 m_2 > 0$ or $m_1 m_2 < 0$ (up to two significant figures)
0.0	1.0	-1.0	0.57
0.1	0.9	-0.8	0.48 or 0.63
0.2	0.8	-0.6	0.44 or 0.64
0.3	0.7	-0.4	0.41 or 0.64
0.4	0.6	-0.2	0.39 or 0.63
0.5	0.5	0.0	0.37
0.6	0.4	0.2	0.36 or 0.60
0.7	0.3	0.4	0.35 or 0.58
0.8	0.2	0.6	0.35 or 0.55
0.9	0.1	0.8	0.36 or 0.51
1.0	0.0	1.0	0.42

Illustration of the table 4.2:

If the blank state is of the form $|\Sigma\rangle = \sqrt{0.1}|0\rangle + \sqrt{0.9}|1\rangle$, then the deletion machine with two transformers delete a qubit with fidelity 0.48 and if the blank state either takes the form $|\Sigma\rangle = -\sqrt{0.1}|0\rangle + \sqrt{0.9}|1\rangle$ or $|\Sigma\rangle = \sqrt{0.1}|0\rangle - \sqrt{0.9}|1\rangle$, then the fidelity of deletion is found to be 0.63.

On the contrary, we observe here that if the product of the parameter of the blank state is negative (i.e. when either m_1 or m_2 is negative), then the deletion machine with two transformers delete the state with fidelity of deletion higher than the case when the product of the parameter of the blank state is positive (i.e. when both m_1 and m_2 are

negative or positive). Also we note that the deletion machine with two transformers deletes a state with fidelity of deletion 0.37 when we use the blank state with parameter $m_1 = m_2 = 0.5$. In addition to this, If we compare the quantum deletion machine with two transformer with the quantum deletion machine with a single transformer, then we find that the deletion machine with a single transformer works better when the product of the amplitudes of the blank state m_1 and m_2 is positive, while the deletion machine with two transformers works better when the product of the parameters m_1 and m_2 is negative.

4.6 PB deleting machine with transformer

In this section, we study the conditional PB deleting machine with the addition of a unitary operator called transformer T. We will show in this section that the addition of a transformer in the quantum deletion machine not only increases the fidelity of deletion but also makes the fidelity of deletion input state independent.

The conditional PB deleting transformation (PB deleter) is defined by equations (1.160-1.163) in chapter-1.

Now if we consider the deletion machine (PB deleter + Transformer) to delete a copy from two copies of the input state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ with $\alpha^2 + |\beta|^2 = 1$ then we find that the final output state from the deletion machine is given by $\rho_{12c}^{out} = T|\psi_d^{out}\rangle_{12c}\langle\psi_d^{out}|T^\dagger$, where $|\psi_d^{out}\rangle_{12c}$ denotes the state after passing through the PB deleter.

The reduced density operator in mode 2 is given by

$$\begin{aligned} \rho_2 = & Tr_{1c}(T|\psi_d^{out}\rangle_{12c}\langle\psi_d^{out}|T^\dagger) = |0\rangle\langle 0|(\frac{\alpha^4 m_1^2}{2} + \frac{\alpha^2 |\beta|^2}{2} + \frac{|\beta|^4 m_1^2}{2} + |\beta|^4 m_2^2) \\ & + |0\rangle\langle 1|(\frac{\alpha^4 m_1 m_2^*}{\sqrt{2}} - \frac{\alpha^2 |\beta|^2}{\sqrt{2}} + \frac{|\beta|^4 m_1 m_2}{\sqrt{2}}) + |1\rangle\langle 0|(\frac{\alpha^4 m_1 m_2}{\sqrt{2}} - \frac{\alpha^2 |\beta|^2}{\sqrt{2}} \\ & + \frac{|\beta|^4 m_1 m_2^*}{\sqrt{2}}) + |1\rangle\langle 1|(\frac{\alpha^4 m_1^2}{2} + \frac{3\alpha^2 |\beta|^2}{2} + \frac{|\beta|^4 m_1^2}{2} + \alpha^4 |m_2|^2) \end{aligned} \quad (4.43)$$

Now to see how well our deleting system deletes a qubit, we have to calculate the fidelity of deletion defined by $F_2 = \langle \Sigma | \rho_2 | \Sigma \rangle$.

If we assume m_1 and m_2 to be real, then

$$\begin{aligned}
 F_2 = & m_1^2 \left[\frac{m_1^2}{2} + \alpha^2 |\beta|^2 \left(\frac{1 - 2m_1^2}{2} \right) + |\beta|^4 m_2^2 \right] + m_1 m_2 \left[\frac{m_1 m_2}{\sqrt{2}} - \alpha^2 |\beta|^2 \left(\frac{1 + 2m_1 m_2}{\sqrt{2}} \right) \right] + \\
 & m_1 m_2 \left[\frac{m_1 m_2}{\sqrt{2}} - \alpha^2 |\beta|^2 \left(\frac{1 + 2m_1 m_2}{\sqrt{2}} \right) \right] + m_2^2 \left[\frac{m_1^2}{2} + \alpha^2 |\beta|^2 \left(\frac{3 - 2m_1^2}{2} \right) + \alpha^4 m_2^2 \right]
 \end{aligned} \tag{4.44}$$

If $m_1 = \frac{1}{\sqrt{2}}$ and $m_2 = -\frac{1}{\sqrt{2}}$, then $F_2 = \frac{1}{2} + \frac{1}{2\sqrt{2}} = 0.85$ (approximately).

Equation (4.44) shows that there exists a blank state for which the fidelity of deletion is input state independent and also its value approaches the optimal cloning fidelity, which we expect from our universal deletion machine. Therefore, the advantage of using the transformer in the quantum deletion machine with PB deleter is that the machine deletes a qubit with fidelity 0.85, which remains the same for all input states. In addition to this, we can observe that the average fidelity of deletion (0.85) for a deletion machine with a PB deleter and transformer is greater than the average fidelity (0.83) for simply a PB deleter.

Chapter 5

Concatenation of quantum cloning and deletion machines

The fundamental concept in social science is Power, in the same sense in which Energy is the fundamental concept in PHYSICS - Bertrand Russell

The idea that time may vary from place to place is a difficult one, but it is the idea Einstein used, and it is correct - believe it or not - Richard Feynman

It is wrong to think that the task of physics is to find out how nature is. Physics concerns what we can say about nature - Niels Bohr

5.1 *Prelude*

The emerging field of quantum computation and Information technology investigated the possibility of exploiting greater information processing ability using qubits [72] . Therefore, manipulation and extraction of quantum information are important tasks in building quantum computer. The copying and deleting of information in a classical computer are inevitable operations whereas similar operations cannot be realized perfectly in quantum computers. Linear evolution makes these quantum operations impossible on arbitrary superpositions of quantum states. Quantum cloning and deleting can both be regarded as devices which perform a unitary operation to distill classical information from quantum information. This doesn't mean that quantum deleting is just the reverse

of quantum cloning. The difference between quantum cloning and quantum deleting can be explained by the following two points: (i) A quantum cloning process can be thought as a swapping operation between the blank qubit and the cloning machine state, but a swapping operation between the cloning state and the deleting machine state can not be thought as a successful deleting process. (ii) The fidelity of each mode is different for the deleting operations whereas it has the same value in the cloning process in symmetric case.

The purpose of this chapter is to find the effect on an arbitrary qubit as a result of the concatenation of quantum cloning and deleting machines. It is a known fact that quantum deleting machine can be applied in a situation when scarcity of memory in a quantum computer occurs. Naturally, a situation may arise in which an arbitrary quantum state is needed to be copied by the imperfect quantum cloner. After performing the given task with cloned qubit if one finds that scarcity of memory occurs then in this situation one has to delete one copy among two cloned copies to store new information in a quantum computer. Consequently in this chapter we will study the concatenation of two quantum operations viz. unitary quantum cloning and deleting transformations. At first, we construct a state dependent quantum deleting machine and show that the minimum average distortion of the input qubit and maximum fidelity of deletion approach to $\frac{1}{3}$ and $\frac{5}{6}$ respectively. Thereafter, we have studied the concatenation of cloning and deletion machines. This chapter is based on our work entitled "Deletion of Imperfect cloned copies" [1].

5.2 State dependent quantum deletion machine

A state dependent quantum deleting transformation can be defined by

$$U|0\rangle|0\rangle|Q\rangle \rightarrow |0\rangle|\Sigma\rangle|A_0\rangle \quad (5.1)$$

$$U|1\rangle|1\rangle|Q\rangle \rightarrow |1\rangle|\Sigma\rangle|A_1\rangle \quad (5.2)$$

$$U|0\rangle|1\rangle|Q\rangle \rightarrow (a_0|0\rangle|1\rangle + b_0|1\rangle|0\rangle)|Q\rangle \quad (5.3)$$

$$U|1\rangle|0\rangle|Q\rangle \rightarrow (a_1|0\rangle|1\rangle + b_1|1\rangle|0\rangle)|Q\rangle \quad (5.4)$$

where $|Q\rangle, |A_0\rangle, |A_1\rangle$ and $|\Sigma\rangle$ have their usual meanings and a_i, b_i ($i=0,1$) are the complex numbers.

Due to the unitarity of the transformation (5.1-5.4) the following relations hold:

$$\langle A_i | A_i \rangle = 1 \quad (i = 0, 1) \quad (5.5)$$

$$|a_i|^2 + |b_i|^2 = 1 \quad (i = 0, 1) \quad (5.6)$$

$$a_i a_{1-i}^* + b_i b_{1-i}^* = 0 \quad (i = 0, 1) \quad (5.7)$$

$$\langle A_1 | Q \rangle = \langle A_0 | Q \rangle = 0. \quad (5.8)$$

Further we assume that

$$\langle A_1 | A_0 \rangle = \langle A_0 | A_1 \rangle = 0 \quad (5.9)$$

A general pure state is given by

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle, \quad \alpha^2 + |\beta|^2 = 1 \quad (5.10)$$

Without any loss of generality we can assume that α and β are real numbers.

Using the transformation relation (5.1-5.4) and exploiting linearity of U, we have

$$\begin{aligned} U|\psi\rangle|\psi\rangle|Q\rangle &= \alpha^2 U|0\rangle|0\rangle|Q\rangle + \alpha\beta U|0\rangle|1\rangle|Q\rangle + \alpha\beta U|1\rangle|0\rangle|Q\rangle + \beta^2 U|1\rangle|1\rangle|Q\rangle \\ &= \alpha^2 |0\rangle|\Sigma\rangle|A_0\rangle + \alpha\beta [g|0\rangle|1\rangle + h|1\rangle|0\rangle]|Q\rangle + \beta^2 |1\rangle|\Sigma\rangle|A_1\rangle \\ &\equiv |\psi\rangle_{12}^{(out)} \end{aligned} \quad (5.11)$$

where $g = a_0 + a_1$, $h = b_0 + b_1$.

The reduced density operators of the output state in mode '1' and '2' are given by

$$\begin{aligned} \rho_1^{(out)} &= \text{tr}_2[\rho_{12}^{(out)}] = \text{tr}_2[|\psi\rangle_{12}^{(out)}\langle\psi|] \\ &= [\alpha^4 + \alpha^2\beta^2 gg^*]|0\rangle\langle 0| + [\beta^4 + \alpha^2\beta^2 hh^*]|1\rangle\langle 1| \end{aligned} \quad (5.12)$$

$$\begin{aligned} \rho_2^{(out)} &= \text{tr}_1[\rho_{12}^{(out)}] = \text{tr}_1[|\psi\rangle_{12}^{(out)}\langle\psi|] \\ &= \alpha^4 |\Sigma\rangle\langle\Sigma| + \alpha^2\beta^2 [gg^*]|1\rangle\langle 1| + hh^*|0\rangle\langle 0| + \beta^4 |\Sigma\rangle\langle\Sigma| \end{aligned} \quad (5.13)$$

Now to see the performance of the machine, we must calculate the distortion of the input state and the fidelity of deletion.

Therefore, the H-S distance between the density operators $\rho_a^{id} = |\psi\rangle\langle\psi|$ and $\rho_1^{(out)}$ given in equation (5.12) is

$$\begin{aligned} D_1(\alpha^2) &= \text{tr}[\rho_1^{(out)} - \rho_a^{id}]^2 \\ &= k\alpha^4\beta^4 + 2\alpha^2\beta^2 \end{aligned} \quad (5.14)$$

where $k = (gg^* - 1)^2 + (hh^* - 1)^2$.

Since D_1 depends on α^2 , so average distortion of input qubit in mode 1 is given by

$$\overline{D_1} = \int_0^1 D_1(\alpha^2) d\alpha^2 = \frac{1}{3} \left(1 + \frac{(gg^* - 1)^2 + (hh^* - 1)^2}{10} \right) \quad (5.15)$$

The reduced density matrix of the qubit in the mode 2 i.e. $\rho_2^{(out)}$ contains error due to imperfect deleting and the error can be measured by calculating the fidelity. Thus the fidelity is given by $F_1 = \langle \Sigma | \rho_2^{(out)} | \Sigma \rangle = 1 - k_1 \alpha^2 \beta^2$, where $k_1 = 2 - gg^* M^2 - hh^* (1 - M^2)$, $M = \langle \Sigma | 1 \rangle$.

Since fidelity of deletion depends on the input state, so the average fidelity over all input state is given by

$$\overline{F_1} = \int_0^1 F_1(\alpha^2) d\alpha^2 = 1 - \frac{k_1}{6} = \frac{2}{3} + \frac{(gg^* - hh^*)M^2 + hh^*}{6} \quad (5.16)$$

From equation (5.15) and (5.16), we observe that the minimum average distortion of the state in mode '1' from the input state is $\frac{1}{3}$ and the minimum average fidelity of deletion is $\frac{2}{3}$. So our prime task is to construct a deleting machine or in other words, to find the value of the machine parameters a_0, a_1, b_0, b_1 which will maximize the fidelity of deletion but keep the average distortion at its minimum value.

To solve the above discussed problem, we take $gg^* - hh^* = \epsilon$ and $hh^* = 1 + \epsilon_1$, where ϵ and ϵ_1 are very small quantities. Then equations (5.15) and (5.16) reduce to

$$\overline{D_1} = \frac{1}{3} + \frac{(\epsilon_1)^2 + (\epsilon + \epsilon_1)^2}{30} \quad (5.17)$$

$$\overline{F_1} = \frac{5}{6} + \frac{\epsilon M^2 + \epsilon_1}{6} \quad (5.18)$$

Therefore, $\overline{D_1} \rightarrow \frac{1}{3}$, $\overline{F_1} \rightarrow \frac{5}{6}$ as $\epsilon, \epsilon_1 \rightarrow 0$.

The above equation shows that if we choose machine parameters a_0, a_1, b_0, b_1 in such a

way that gg^* and hh^* both are very close to unity then only we are able to keep the distortion at its minimum level and increase the average fidelity to $\frac{5}{6}$.

5.3 Concatenation of cloning and deletion machines

In this section, we study the effect of concatenation of cloning and deleting machine. We investigate how well one can delete one copy from the two imperfectly cloned copies of an unknown quantum state. We consider here only the imperfect cloned copies obtained from WZ cloning machine and BH cloning machine.

5.3.1 Concatenation of WZ cloning machine and PB deleting machine

Let an unknown quantum state (5.10) be cloned by WZ cloning machine. Using cloning transformation (1.17-1.18), an unknown quantum state (5.10) cloned to

$$\alpha|0\rangle|0\rangle|Q_0\rangle + \beta|1\rangle|1\rangle|Q_1\rangle \quad (5.19)$$

Now, operating deleting machine (4.1-4.4) on the cloned state (5.19), we get the final output state as

$$|\phi\rangle_{xy}^{(out)} = \alpha|0\rangle|\Sigma\rangle|A_0\rangle + \beta|1\rangle|\Sigma\rangle|A_1\rangle \quad (5.20)$$

The reduced density operator describing the output state in modes x and y are given by

$$\rho_x^{(out)} = tr_y(\rho_{xy}) = \alpha^2|0\rangle\langle 0| + \beta^2|1\rangle\langle 1| \quad (5.21)$$

$$\rho_y^{(out)} = tr_x(\rho_{xy}) = |\Sigma\rangle\langle \Sigma| \quad (5.22)$$

The H-S distance between the density operators $\rho_a^{id} = |\psi\rangle\langle \psi|$ and $\rho_x^{(out)}$ given in equation (5.21) is

$$D_3(\alpha^2) = tr[\rho_x^{(out)} - \rho_a^{id}]^2 = 2\alpha^2(1 - \alpha^2) \quad (5.23)$$

The average distortion of input qubit after cloning and deleting operation is given by

$$\overline{D_3} = \int_0^1 D_3(\alpha^2) d\alpha^2 = 0.33 \quad (5.24)$$

The fidelity of deletion is given by

$$F_3 = \langle \Sigma | \rho_y | \Sigma \rangle = 1 \quad (5.25)$$

The above equations shows that if we clone an unknown quantum state by WZ cloning machine, and delete a copy qubit by Pati and Braunstein's deleting machine then the fidelity of deletion is found to be 1 for arbitrary input state but the concatenation of the cloning and deleting machine cannot retain the input qubit in its original state.

5.3.2 Concatenation of BH cloning machine and PB deleting machine

Let an unknown quantum state (5.10) be cloned by B-H cloning machine. Using cloning transformation (1.48-1.49), quantum state (5.10) is cloned to

$$\alpha[|0\rangle|0\rangle|Q_0\rangle + (|0\rangle|1\rangle + |1\rangle|0\rangle)|Y_0\rangle] + \beta[|1\rangle|1\rangle|Q_1\rangle + (|0\rangle|1\rangle + |1\rangle|0\rangle)|Y_1\rangle] \quad (5.26)$$

After operating deleting machine (4.1-4.4) to the cloned state (5.26), the output state is given by

$$|\phi\rangle_{xy}^{(out)} = \frac{1}{\sqrt{1+2\xi}} \{ \alpha[|0\rangle|\Sigma\rangle|A_0\rangle + (|0\rangle|1\rangle + |1\rangle|0\rangle)|Y_0\rangle] + \beta[|1\rangle|\Sigma\rangle|A_1\rangle + (|0\rangle|1\rangle + |1\rangle|0\rangle)|Y_1\rangle] \} \quad (5.27)$$

The reduced density operators describing the output state in mode x and y are given by

$$\rho_x^{(out)} = tr_y(\rho_{xy}) = tr_y(|\phi\rangle_{xy}^{(out)}\langle\phi|) = \frac{1}{1+2\xi} \{ (\alpha^2 + \xi)|0\rangle\langle 0| + (\beta^2 + \xi)|1\rangle\langle 1| \} \quad (5.28)$$

$$\rho_y^{(out)} = tr_x(\rho_{xy}) = tr_y(|\phi\rangle_{xy}^{(out)}\langle\phi|) = \frac{1}{1+2\xi} \{ |\Sigma\rangle\langle\Sigma| + I\xi \} \quad (5.29)$$

where I is the identity matrix in two dimensional Hilbert space.

The distance between the density operators $\rho_a^{id} = |\psi\rangle\langle\psi|$ and $\rho_x^{(out)}$ given in equation (5.28) is

$$D_4(\alpha^2) = tr[\rho_x^{(out)} - \rho_a^{id}]^2 = \frac{2\xi^2 + 2\alpha^2\beta^2(1+4\xi)}{(1+2\xi)^2} \quad (5.30)$$

The average distortion of input state is given by

$$\overline{D_4} = \int_0^1 D_4(\alpha^2) d\alpha^2 = \frac{6\xi^2 + 4\xi + 1}{3(1+2\xi)^2} = \frac{11}{32}, \text{ for B-H cloning machine } \xi = \frac{1}{6} \quad (5.31)$$

Also the fidelity of deletion of one qubit from two identical cloned qubits is given by

$$F_4 = \langle \Sigma | \rho_y | \Sigma \rangle = \frac{1 + \xi}{1 + 2\xi} = \frac{7}{8}, \text{ for B-H cloning machine } \xi = \frac{1}{6} \quad (5.32)$$

5.3.3 Concatenation of WZ cloning machine and deleting machine(5.1-5.4)

After operating deleting machine (5.1-5.4) on the cloned state (5.19), we get the output state as

$$|\phi\rangle_{xy}^{(out)} = \alpha|0\rangle|\Sigma\rangle|A_0\rangle + \beta[|1\rangle|\Sigma\rangle|A_1\rangle] \quad (5.33)$$

The reduced density operators describing the output state in mode x and y are given by

$$\rho_x^{(out)} = \text{tr}_y(\rho_{xy}) = \text{tr}_y(|\phi\rangle_{xy}^{(out)}\langle\phi|) = \alpha^2|0\rangle\langle 0| + \beta^2|1\rangle\langle 1| \quad (5.34)$$

$$\rho_y^{(out)} = \text{tr}_x(\rho_{xy}) = \text{tr}_x(|\phi\rangle_{xy}^{(out)}\langle\phi|) = |\Sigma\rangle\langle\Sigma| \quad (5.35)$$

The H-S distance between the density operators $\rho_a^{id} = |\psi\rangle\langle\psi|$ and $\rho_x^{(out)}$ given in equation (5.34) is

$$D_5(\alpha^2) = \text{tr}[\rho_x^{(out)} - \rho_a^{id}]^2 = 2\alpha^2(1 - \alpha^2) \quad (5.36)$$

Since D_5 depends on α^2 , so average distortion of deletion is given by

$$\overline{D_5} = \int_0^1 D_5(\alpha^2) d\alpha^2 = 0.33 \quad (5.37)$$

The fidelity of the second qubit is given by

$$F_5 = \langle \Sigma | \rho_y | \Sigma \rangle = 1 \quad (5.38)$$

5.3.4 Concatenation of BH cloning machine and deleting machine(5.1-5.4)

After operating deleting machine (5.1-5.4) on the cloned state (5.26), we get

$$\begin{aligned} |\phi\rangle_{xy}^{(out)} = & \{ \alpha[|0\rangle|\Sigma\rangle|A_0\rangle + (g|0\rangle|1\rangle + h|1\rangle|0\rangle)|Y_0\rangle] + \beta[|1\rangle|\Sigma\rangle|A_1\rangle + \\ & (g|0\rangle|1\rangle + h|1\rangle|0\rangle)|Y_1\rangle] \} \end{aligned} \quad (5.39)$$

We assume

$$\langle A_0|Y_0\rangle = \langle A_1|Y_1\rangle = 0 \quad (5.40)$$

The reduced density operators describing the output state in two different modes are given by

$$\begin{aligned} \rho_x^{(out)} &= \text{tr}_y(\rho_{xy}) = \text{tr}_y(|\phi\rangle_{xy}^{(out)}\langle\phi|) \\ &= \frac{1}{1 + (gg^* + hh^*)\xi} \{(\alpha^2 + \xi gg^*)|0\rangle\langle 0| + (\beta^2 + \xi hh^*)|1\rangle\langle 1|\} \end{aligned} \quad (5.41)$$

$$\begin{aligned} \rho_y^{(out)} &= \text{tr}_x(\rho_{xy}) = \text{tr}_x(|\phi\rangle_{xy}^{(out)}\langle\phi|) \\ &= \frac{1}{1 + (gg^* + hh^*)\xi} \{|\Sigma\rangle\langle\Sigma| + (\xi hh^*)|0\rangle\langle 0| + (\xi gg^*)|1\rangle\langle 1|\} \end{aligned} \quad (5.42)$$

Now in order to measure the degree of distortion, we evaluate the distance between the density operators $\rho_a^{id} = |\psi\rangle\langle\psi|$ and $\rho_x^{(out)}$ given in equation (5.41) is given by

$$D_6(\alpha^2) = \text{tr}[\rho_x^{(out)} - \rho_a^{id}]^2 = \frac{2\xi^2(gg^*\beta^2 - hh^*\alpha^2)^2}{[1 + (gg^* + hh^*)\xi]^2} \quad (5.43)$$

The average distortion of input qubit is given by

$$\begin{aligned} \overline{D_6} &= \int_0^1 D_6(\alpha^2) d\alpha^2 = \frac{1}{3} + \frac{2\xi^2[(gg^*)^2 + (hh^*)^2 - (gg^*)(hh^*)]}{3[1 + (gg^* + hh^*)\xi]^2} \\ &= \frac{1}{3} + \frac{2}{3} \left(\frac{(gg^*)^2 + (hh^*)^2 - (gg^*)(hh^*)}{[6 + gg^* + hh^*]^2} \right), \text{ for } \xi = \frac{1}{6} \end{aligned} \quad (5.44)$$

The fidelity of deletion is given by

$$\begin{aligned} F_6 &= \langle\Sigma|\rho_y|\Sigma\rangle = \frac{1 + \xi M^2(gg^* - hh^*) + \xi(hh^*)}{1 + (gg^* + hh^*)\xi} \\ &= \frac{6 + M^2(gg^* - hh^*) + (hh^*)}{6 + gg^* + hh^*}, \text{ for B-H cloning machine } \xi = \frac{1}{6} \end{aligned} \quad (5.45)$$

In particular, For $a_0 = \frac{\sqrt{3}}{2}, a_1 = \frac{i}{2}, b_0 = \frac{i}{2}, b_1 = \frac{\sqrt{3}}{2}$, we get $gg^* = hh^* = 1$. In this case, we find that the fidelity of deletion and the average distortion is same as in the case of B-H cloning machine and P-B deleting machine.

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Journal Ref.: Physica Scripta **74**, 555 (2006).

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Bibliography

- [1] Adhikari, S., B.S.Choudhury, J. Phys. A: Math. Gen. **37**, 1 (2004).
- [2] Adhikari, S., Phys. Rev. A **72**, 052321 (2005).
- [3] Adhikari, S., B.S.Choudhury, I.Chakrabarty, J. Phys. A: Math. Gen. **39**, 8439 (2006).
- [4] Adhikari, S., B.S.Choudhury, Phys. Rev. A **73**, 054303 (2006).
- [5] Adhikari, S., B.S.Choudhury, Phys. Rev. A **74**, 032323 (2006).
- [6] Azuma, K., J.Shimamura, M.Koashi and N.Imoto, Phys. Rev. A **72**, 032335 (2005).
- [7] Audenaert, K., F.Verstraete, T.D.Bie and B.D.Moor, e-print quant-ph/0012074.
- [8] Audenaert, K.M.R and J.Eisert, J. Math. Phys. **46**, 102104 (2005).
- [9] Bandyopadhyay, S. and G.Kar, Phys. Rev. A **60**, 3296 (1999).
- [10] Barnum, H., C.M.Caves, C.A.Fuchs, R.Jozsa and B.Schumacher, Phys. Rev. Lett. **76**, 2818 (1996).
- [11] Bennett, C.H., and G.Brassard, Proceedings of IEEE International Conference on Computers, System and Signal Processing, Bangalore, India, 1984, pp.175-179.
- [12] Bennett, C.H., and S.J.Weisner, Phys. Rev. Lett. **69**, 2881 (1992).
- [13] Bennett, C.H., Phys. Rev. Lett. **68**, 3121 (1992).
- [14] Bennett, C.H., G.Brassard, C.Crepeau, R.Jozsa, A.Peres and W.K.Wootters, Phys. Rev. Lett. **70**, 1895 (1993).

- [15] Bennett, C.H., H.J.Bernstein, S.Popescu and B.Schumacher, Phys. Rev. A **53**, 2046 (1996).
- [16] Bennett, C.H., D.P.DiVincenzo, J.A.Smolín and W.K.Wootters, Phys. Rev. A **54**, 3824 (1996).
- [17] Bose, S., V.Vedral and P.L.Knight, Phys. Rev. A **57**, 822 (1998).
- [18] Braunstein, S.L., V.Buzek and M.Hillery, Phys. Rev. A **63**, 052313 (2001).
- [19] Bruss, D., A.Ekert and C.Macchiavello, Phys. Rev. Lett. **81**, 2598 (1998).
- [20] Bruss, D., D.P.DiVincenzo, A.Ekert, C.A.Fuchs, C.Macchiavello and J.A.Smolín, Phys. Rev. A **57**, 2368 (1998).
- [21] Bruss, D. and C.Macchiavello, Phys. Lett. A **253**, 249 (1999).
- [22] Bruss, D., G.M.D'Ariano, C.Macchiavello, and M.F.Sacchi, Phys. Rev. A **62**, 62302 (2000).
- [23] Bruss, D., J.Calsamiglia and N.Lutkenhaus, Phys. Rev. A **63**, 042308 (2001).
- [24] Bruss, D., M.Cinchetti, G.M.D'Ariano and C.Macchiavello, Phys. Rev. A **62**, 12302 (2000).
- [25] Bruss, D. and C.Macchiavello, J. Phys. A: Math. Gen. **34**, 6815 (2001).
- [26] Bruss, D. and C.Macchiavello, Found. Phys.**33**, 1617 (2003).
- [27] Buscemi, F., G.M.D'Ariano and C.Macchiavello, Phys. Rev. A **71**, 042327 (2005).
- [28] Buzek, V., and M. Hillery, Phys. Rev. A **54**, 1844 (1996).
- [29] Buzek, V., V.Vedral, M.B.Plenio, P.L.Knight and M.Hillery, Phys. Rev. A **55**, 3327 (1997).
- [30] Buzek, V., S.L.Braunstein, M.Hillery, D.Bruss, Phys. Rev. A **56**, 3446 (1997).
- [31] Buzek, V., and M. Hillery, Phys. Rev. Lett. **81**, 5003 (1998).

- [32] Buzek, V., M. Hillery, and R. Bednik, *Acta Phys. Slov.* **48**, 177 (1998).
- [33] Buzek, V., M. Hillery and R. F. Werner, *Phys. Rev. A* **60**, R2626 (1999).
- [34] Cabello, A., *Phys. Rev. A* **61**, 052312 (2000).
- [35] Cerf, N.J., *Acta Phys. Slov.* **48**, 115 (1998).
- [36] Cerf, N.J., *Phys. Rev. Lett.* **84**, 4497 (2000).
- [37] Cerf, N.J., *J. Mod. Opt.* **47**, 187 (2000).
- [38] Cerf, N.J., T. Durt, and N. Gisin, *J. Mod. Opt.* **49**, 1355 (2002).
- [39] Cerf, N.J., and J. Fiurasek, e-print quant-ph/0512172.
- [40] Chefles, A. and S.M. Barnett, *Phys. Rev. A* **60**, 136 (1999).
- [41] Chen, K., S. Alberverio and S-M Fei, *Phys. Rev. Lett.* **95**, 210501 (2005).
- [42] Chiara, G.D., R. Fazio, C. Macchiavello, S. Montangero and G.M. Palma, *Phys. Rev. A* **70**, 062308 (2004).
- [43] Chiara, G.D., R. Fazio, C. Macchiavello, S. Montangero and G.M. Palma, *Phys. Rev. A* **72**, 012328 (2005).
- [44] Chiribella, G., G.M. D'Ariano, and P. Perinotti, *Phys. Rev. A* **72**, 042336 (2005).
- [45] D'Ariano, G.M., and H.P. Yuen, *Phys. Rev. Lett.* **76**, 2832 (1996).
- [46] D'Ariano, G.M., and P.L. Presti, *Phys. Rev. A* **64**, 042308 (2001).
- [47] Deutsch, D., A. Ekert, R. Jozsa, C. Macchiavello, S. Popescu and A. Sanpera, *Phys. Rev. Lett.* **77**, 2818 (1996).
- [48] Demkowicz-dobrzanski, R., M. Lewenstein, A. Sen(De), U. Sen, and D. Bruss, *Phys. Rev. A* **73**, 032313 (2006).
- [49] Derka, R., V. Buzek and A. Ekert, *Phys. Rev. Lett.* **80**, 1571 (1998).

- [50] Duan, L.M., and G.C.Guo, Phys. Rev. Lett. **80**, 4999 (1998).
- [51] Duan, L.M., G-C Guo, Phys. Rev. A **64**, 054101 (2001).
- [52] Duan, L.M., G-C Guo, e-print quant-ph/9704020.
- [53] Durt, T., and J.Du, Phys. Rev. A **69**, 062316 (2004).
- [54] Durt, T., J.Fiurasek, and N.J.Cerf, Phys. Rev. A **72**, 052322 (2005).
- [55] Ekert, A., Phys. Rev. Lett. **67**, 661 (1991).
- [56] Einstein, A., B.Podolsky, and N.Rosen, Phys. Rev. **47**, 777 (1935).
- [57] Eisert, J. and M.B.Plenio, J. Mod. Opt. **46**, 145 (1999).
- [58] Fan, H., X-B Wang, and K.Matsumoto, e-print quant-ph/0012033.
- [59] Fan, H., K.Matsumoto, and M.Wadati, Phys. Rev. A **64**, 064301 (2001).
- [60] Fan, H., K.Matsumoto, X-B Wang, and M.Wadati, Phys. Rev. A **65**, 012304 (2002).
- [61] Fan, H., H.Imai, K.Matsumoto, and X-B Wang, Phys. Rev. A **67**, 022317 (2003).
- [62] Fan, H., Phys. Rev. A **68**, 054301 (2003).
- [63] Fan, H., B.Liu, and K.Shi, e-print quant-ph/0601017.
- [64] Fei, S-M., Z-X Wang and H.Zhao, Phys. Lett. A **329**, 414 (2004).
- [65] Feng, J., Y-F Gao, J-S Wang, and M-S Zhan, Phys. Rev. A **65**, 052311 (2002).
- [66] Filip, R., Phys. Rev. A **65**, 062320 (2002)
- [67] Filip, R., e-print quant-ph/0401036.
- [68] Fiurasek, J., S.Iblisdir, S.Massar, and N.J.Cerf, Phys. Rev. A **65**, 040302(R) (2002).
- [69] Fiurasek, J., Phys. Rev. A **70**, 032308 (2004).
- [70] Fiurasek, J., R.Filip, and N.J.Cerf, Quan. Inf. Comp. **5**, 583 (2005).

- [71] Fuchs, C., N.Gisin, R.B.Griffiths, C.S.Niu, and A.Peres, Phys. Rev. A **56**, 1163 (1997).
- [72] Galvao,F.E, and L.Hardy, Phys. Rev. A **62**, 022301 (2000).
- [73] Ghirardi, G., A.Rimini, and T.Weber, Lett. Nuovo Cimento **27**, 293 (1980).
- [74] Ghirardi, G., R.Grassi, A.Rimini, and T.Weber, Europhys. Lett. **6**, 95 (1988).
- [75] Ghiu, I., Phys. Rev. A **67**, 012323 (2003).
- [76] Ghosh, S., G.Kar, and A.Roy, Phys. Lett. A **261**, 17 (1999).
- [77] Ghosh, S., G.Kar, S.Kunkri and A.Roy, e-print quant-ph/0312045.
- [78] Gisin, N., Phys. Lett. A **154**, 201 (1991).
- [79] Gisin, N., and S. Massar, Phys. Rev. Lett. **79**, 2153 (1997).
- [80] Gisin, N., and B. Hutner, Phys. Lett. A **228**, 13 (1997).
- [81] Gisin, N., Phys. Lett. A **242**, 1 (1998).
- [82] Gisin, N., and S. Popescu, Phys. Rev. Lett. **83**, 432 (1999).
- [83] Goldenberg, L., and L.Vaidman, Phys. Rev. Lett. **75**, 1239 (1995).
- [84] Gorbachev, V.N., A.I.Trubilko, A.A.Rodichkina, Phys. Lett. A **314** (2003) 267.
- [85] Greenberger, D.M., M.A.Horne, A.Shimony, and A.Zeilinger, Amer. J. Phys. **58**, 1131 (1990).
- [86] Hardy, L., and D. D. Song, Phys. Lett. A **259**, 331 (1999).
- [87] Han, Y-J., Y-S Zhang and G.C.Guo, e-print quant-ph/0209094.
- [88] Herbert, N., Found. Phys. **12**, 1171 (1982).
- [89] Hillery, M., and V.Buzek, Phys. Rev. A **56**, 1212 (1997).
- [90] Horodecki, M., R. Horodecki, A. Sen(De) and U. Sen, e-print quant-ph/0306044.

- [91] Horodecki, M., P.Horodecki, R.Horodecki, Phys. Lett. A **223**, 1 (1996).
- [92] Horodecki, M., P.Horodecki, R.Horodecki, Phys. Rev. Lett. **80**, 5239 (1998).
- [93] Horodecki, M., P.Horodecki, R.Horodecki and M.Piani, e-print quant-ph/0506174.
- [94] Iblisdir, S., A.Acin, N.J.Cerf, R.Filip, J.Fiurasek and N.Gisin, Phys. Rev. A **72**, 042328 (2005).
- [95] Iblisdir, S., A.Acin, and N.Gisin, e-print quant-ph/0505152.
- [96] Ishizaka, S. and T.Hiroshima, Phys. Rev. A **62**, 022310 (2000).
- [97] Jozsa, R., e-print quant-ph/0204153.
- [98] Karimipour, V., and A.T.Rezakhani, Phys. Rev. A **66**, 052111 (2002).
- [99] Keyl, M., and R.F.Werner, J. Math. Phys. **40**, 3283 (1999).
- [100] Kent, A., N.Linden and S.Massar, Phys. Rev. Lett. **83**, 2656 (1999).
- [101] Kwek, L. C., C.H.Oh, X-B Wang and Y.Yeo, Phys. Rev. A **62**, 052313 (2000).
- [102] Lamoureaux, L.-P., and N.J.Cerf, Quan. Inf. Com. **5**, 032 (2005).
- [103] Landauer, R., Physics Today **44(5)**, 22 (1991).
- [104] Lee, S., D.P.Chi, S.D.Oh and J.Kim, Phys. Rev. A **68**, 062304 (2003).
- [105] Li, C., H-S Song and L.Zhou, Journal of Optics B: Quantum semiclass. opt. **5**, 155 (2003).
- [106] Maruyama, K., and P.L. Knight, e-print quant-ph/0309167.
- [107] Masiak, P., and P.L. Knight, Fortsch. Phys. **49**, 1001 (2001).
- [108] Massar,S., S.Popescu, Phys. Rev. Lett. **74**, 1259 (1995).
- [109] Miranowicz, A. and A.Grudka, J. Opt. B: Quantum Semiclass. Optics **6**, 542 (2004).

- [110] Navez, P., and N.J.Cerf, Phys. Rev. A **68**, 032313 (2003).
- [111] Nielsen, M.A., and I.L.Chuang, Cambridge University Press, Cambridge, 2000.
- [112] Niu, C-S., and R.B.Griffiths, Phys. Rev. A **58**, 4377 (1998).
- [113] Niu, C-S., and R.B.Griffiths, Phys. Rev. A **60**, 2764 (1999).
- [114] Osborne, T.J., Phys. Rev. A **72**, 022309 (2005).
- [115] Ozawa, M., Phys. Lett. A **268**, 158 (2000).
- [116] Pati, A. K., Phys. Rev. Lett. **83**, 2849 (1999).
- [117] Pati, A. K., Phys. Lett. A **270**, 103 (2000).
- [118] Pati, A. K., and S. L. Braunstein, Nature **404**, 164 (2000).
- [119] Pati, A. K., and S. L. Braunstein, e-print quant-ph/0007121.
- [120] Pati, A. K., Phys. Rev. A **66**, 062319 (2002).
- [121] Pati, A. K., and S. L. Braunstein, Phys. Lett. A **315**, 208 (2003).
- [122] Pati, A. K., Fluc. and Noise Lett. **4**, R27 (2004).
- [123] Peres, A., Phys. Rev. Lett. **77**, 1413 (1996).
- [124] Peres, A., Phys. Rev. A **61**, 022117 (2000).
- [125] Plenio, M.B. and V.Vedral, Contemp. Phys. **39**, 431 (1998).
- [126] Qiu, D., Phys. Lett. A **301**, 112 (2002).
- [127] Qiu, D., Phys. Rev. A **65**, 052329 (2002).
- [128] Qiu, D., Phys. Lett. A **308**, 335 (2003).
- [129] Rastegin, A.E., e-print quant-ph/0208159.
- [130] Rastegin, A.E., Phys. Rev. A **67**, 012305 (2003).

- [131] Rastegin, A.E., Phys. Rev. A **68**, 032303 (2003).
- [132] Roy, A., A.Sen(De) and U.Sen, Phys. Lett. A **286**, 1 (2001).
- [133] Scarani, V., S.Iblisdir and N.Gisin, Rev. Mod. Phys. **77**, 1225 (2005).
- [134] Scherer, H., and P.Busch, Phys. Rev. A **47**, 1647 (1993).
- [135] Shor, P.W., and J.Preskill, Phys. Rev. Lett. **85**, 441 (2000).
- [136] Song, D.D., and L.Hardy, e-print quant-ph/0001105.
- [137] Song, W., M.Yang and Z-L Cao, Phys. Lett. A **327**, 123 (2004).
- [138] Ting, G., Y.Feng-Li and W.Zhi-Xi, Chinese Phys.Lett. **21**, 995 (2004).
- [139] Vedral.V, M.B.Plenio, M.A.Rippin and P.L.Knight, Phys. Rev. Lett. **78**, 2275 (1997).
- [140] Vedral.V and M.B.Plenio, Phys. Rev. A **57**, 1619 (1998).
- [141] Vidal. G and R.Tarrach, Phys. Rev. A **59**, 141 (1999).
- [142] Vidal. G and R.F.Werner, Phys. Rev. A **65**, 032314 (2002).
- [143] Verstraete, F., K.Audenaert, J.Dehaene and B.D.Moor, J. Phys. A: Math. Gen. **34**, 10327 (2001).
- [144] Werner, R.F., Phys. Rev. A **40**, 4277 (1989).
- [145] Werner, R.F., Phys. Rev. A **58**, 1827 (1998).
- [146] Wigner, E.P., The logic of personal knowledge: Essays presented to Michael Polany on his Seventieth birthday, (Routledge and Kegan Paul, London) **231** (1961).
- [147] Wootters, W.K., and W. H. Zurek, Nature **299**, 802 (1982).
- [148] Wootters, W.K., Phys. Rev. Lett. **80**, 2245 (1998).
- [149] Ying, M., Phys. Lett. A **299**, 107 (2002).

- [150] Ying, M., Phys. Lett. A **302**, 1 (2002).
- [151] Yuen, H.P., Phys. Lett. A **113**, 405 (1986).
- [152] Zanardi, P., Phys. Rev. A **58**, 3484 (1998).
- [153] Zhang, C-W., Z-Y Wang, C-F Li and G-C Guo, Phys. Rev. A **61**, 062310 (2000).
- [154] Zhang, C-W., C-F Li and G-C Guo, Phys. Lett. A **271**, 31 (2000).
- [155] Zhang, C-W., C-F Li and G-C Guo, e-print quant-ph/9908002.
- [156] Zhang, C-W., C-F Li, Z-Y Wang and G-C Guo, Phys. Rev. A **62**, 042302 (2000).
- [157] Zhou, D., B. Zeng, and L. You, Phys. Lett. A **352**, 41 (2006).
- [158] Zukowski, M., A. Zeilinger, M.A. Horne and A.K. Ekert, Phys. Rev. Lett. **71**, 4287 (1993).
- [159] Zurek, W. H., Nature **404**, 130 (2000).